B3. Perturbation Expansions & Feynman Diagrams

We now come to a very important part of the apparatus of QFT (as well as of classical field theory, statistical mechanics, and indeed ordinary mechanics), viz., the use of perturbation expansions and their associated Feynman diagram expansions to get approximate results for quantities.

This is a big topic, and we cannot possibly cover all of it. So I will be picking certain important things and focussing on them, and only briefly discussing others. As before, the main ideas will be developed using ϕ^4 theory to begin with, since this is a relatively simple bosonic field theory with interactions. We then move on to the Feynman rules for Dirac fermions, non-relativistic phonons, and for non-relativistic fermions - this should give a fair picture of how this works.

It is important to see how these rules translate into concrete calculations. We therefore take a much closer look at one special case, that of phonons - we look at these for both solids and liquids. This example is of great importance in condensed matter physics, and it also brings out the role of symmetries, conservation laws, and other general features which appear in many field theories.

Finally, we look at a system of coupled fields. Again, this illustrates a number of crucial general features of QFT - given its importance we first treat a toy model of 2 coupled relativistic scalar fields, and then go on to discuss the very important non-relativistic example of the coupled electron-phonon system. This is done first for metallic electrons, where this interaction leads to superconductivity, and then for insulators, where it leads to the formation of polarons.

There are many important general structural features of QFT which can be discerned already in perturbation theory; I will delineate some of these, taking care to note that perturbative results can also be very misleading. We will return to this topic again later on, in discussing non-perturbative phenomena (see Chapters 5 and 6).

B.3.1: DIAGRAMMATIC EXPANSIONS for ϕ^4 THEORY

In the last chapter we discussed the basic quantities of interest in a QFT, using ϕ^4 theory as our main example. These quantities included the 3 generating functionals $\mathcal{Z}[J]$, $W[J]$, and $\Gamma[\phi]$,and their associated correlation functions $G_n(x_1,\ldots,x_n)$, $\mathcal{G}_n^{(c)}(x_1,\ldots,x_n)$, and $\Gamma_n(x_1,\ldots,x_n)$.

In this section we will discuss how to evaluate these quantities using perturbative expansions in powers of the coupling constant g. As we will see later on, this is by no means

the only parameter in which one can expand, which can be then be used to generate whole classes of diagrams. Later on (in Chapter 6) we will meet "gradient expansions" (when at least some part of the field is varying slowly in x-space), "loop" expansions (also called fluctuation expansions, or asymptotic expansions in $\hbar$), and $1/N$ expansions (where N is the number of components of the field); and so on.

Our two main tools, at least initially, for the generation of perturbative expansions will be the expressions eqs. (79) and (83) of Chapter B.1; for the ϕ^4 theory these were given in eqs. (80) and (84) of Chapter B.1. Let's begin with the expansion generated by functional derivatives in $J(x)$, which we repeat here:

$$\mathcal{Z}_0[J] \;=\; \frac{e^{-\frac{i}{\hbar}\int d^Dx\, V\left(-i\hbar\frac{\delta}{\delta J(x)}\right)}\; e^{-\frac{i}{2\hbar}\int d^Dx\,\int d^Dx' J(x)\Delta_F(x-x')J(x')}}{\left[e^{-\frac{i}{\hbar}\int d^Dx\, V\left(-i\hbar\frac{\delta}{\delta J(x)}\right)}\; e^{-\frac{i}{2\hbar}\int d^Dx\,\int d^Dx' J(x)\Delta_F(x-x')J(x')}\right]\,|_{J=0}} \tag{1}$$

where as usual we choose the interaction

$$V[\phi] \;=\; \int d^Dx\,\frac{g}{4!}\phi^4(x) \tag{2}$$

We will also deal with the related functions

$$\begin{aligned} W[J] &= -i\hbar Z[J] \\ \Gamma[\phi] &= W[J] - \int d^Dx\, J(x)\phi(x) \end{aligned} \tag{3}$$

and the task we are setting ourselves here is to determine the perturbative expansion in g of these functions, along with their associated functional derivatives (i.e., their associated correlators). So let's begin.

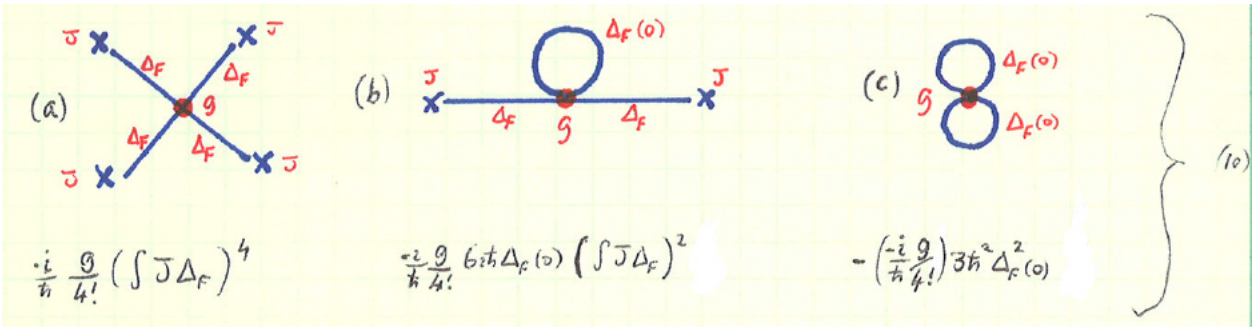
B.3.1(a) : EXPANSION of $\mathcal{Z}[J]$ and $G_n(x_1,\ldots,x_n)$ for ϕ^4 THEORY

Let's do this explicitly, so that there is no confusion about what is involved. From eq. (1) it is clear that we want to calculate the quantity

$$I_4[J] \;=\; \big(-i\hbar\frac{\delta}{\delta J(x)}\big)^4\; e^{-\frac{i}{2\hbar}\int d^Dx_1\,\int d^Dx_2\; J(x)\Delta_F(x_1-x_2)J(x_2)} \tag{4}$$

in order to find the first correction (to order g) in $\mathcal{Z}[J]$ from the non-interacting expression $\mathcal{Z}_0[J]$. This is straightforward; for example, we see that

$$\begin{aligned} &-i\hbar\frac{\delta}{\delta J(x)}\left[e^{-\frac{i}{2\hbar}\int d^Dx_1\,\int d^Dx_2 J(x)\Delta_F(x_1-x_2)J(x_2)}\right] \\ &= \;-\hbar\int d^Dx_1\, J(x_1)e^{-\frac{i}{2\hbar}\int d^Dx_1\,\int d^Dx_2 J(x)\Delta_F(x_1-x_2)J(x_2)} \end{aligned} \tag{5}$$



and continuing in the fashion, we get

$$I_4[J] \;\rightarrow\; \left\{\left[\int d^Dx'\, J(x')\Delta_F(x-x')\right]^4 + 6i\hbar\Delta_F(0)\left[\int d^Dx'\Delta_F(x-x')J(x')\right]^2 - 3\hbar^2\Delta_F^2(0)\right\} \times e^{-\frac{i}{2\hbar}\int d^Dx_1\int d^Dx_2 J(x)\Delta_F(x_1-x_2)J(x_2)} \tag{6}$$

and the $J(x)\rightarrow 0$ limit of this is

$$\lim_{J(x)\rightarrow 0} I_4[J] \;=\; -3\hbar^2\Delta_F^2(0) \tag{7}$$

We then find that, to order g,

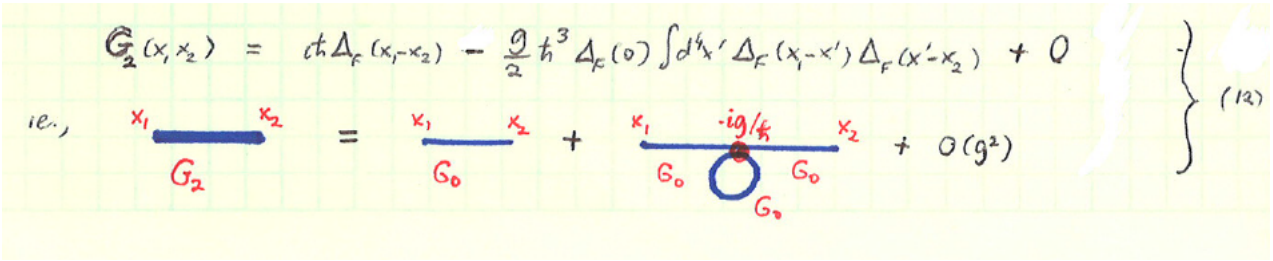
$$\mathcal{Z}[J] \;=\; \frac{\left\{1-\frac{ig}{\hbar 4!}\left[\left(\int J\Delta_F\right)^4 + 6i\hbar\Delta_F(0)\left(\int J\Delta_F\right)^2 - 3\hbar^2\Delta_F^2(0)\right]\right\} e^{\frac{-i}{2\hbar}\iint J\Delta_F J}}{1+\frac{ig}{\hbar 4!}\left(3\hbar^2\Delta_F^2(0)\right)} + \mathcal{O}(g^2) \tag{8}$$

$$= \left\{1-\frac{ig}{\hbar 4!}\left[\left(\int J\Delta_F\right)^4 + 6i\hbar\Delta_F(0)\left(\int J\Delta_F\right)^2\right]\right\} e^{\frac{-i}{2\hbar}\iint J\Delta_F J} + \mathcal{O}(g^2) \tag{9}$$

in an obvious abbreviated notation. Clearly this is quite elaborate - but it has a simple interpretation in terms of diagrams. Suppose we use the same diagrammatic notation as we used in 1st section, in eq. B.1 (66) (section B.1, page 12) so that we can represent the key terms in eq. (8) above as in the figure showing eqtn. (10)

Let us note 2 important features that appear already in eqs. (8)-(10); where we note that the diagrams in (10) are those that appear in the numerator of eq. (8).

(i) There are 3 different kinds of diagram appearing already in (10). The first graph, at left, is an example of a diagram which contributes to scattering in dynamics of the ϕ^4 field. It is a "4-point" function, i.e., it has 4 external legs; it also gives an irreducible contribution (i.e., are that cannot be separated into 2 parts by cutting any internal lines) to the correlator $G_4(x_1,\ldots,x_4)$. You should look back at eq. B.1 (22) in this connection, and at eq. B.1 (18). The 2nd, middle diagram in (10) is actually a contribution to $G_2(x_1,x_2)$, and it contains within it the self-energy part $i\hbar g\Delta_F(0)/4$; a contribution to $-i\Sigma(k)$ (see B.1 (47)). Thus we see that the 2nd diagram is also a contribution to the connected Green function $\mathcal{G}_2^{(c)}(x_1,x_2)$



in eq. B.1 (48). The third diagram, right in (10), is a new kind of "vacuum diagram", or "*vacuum fluctuation*" diagram, which contributes to the ground-state energy of the system - we shall say more on these later.

(ii) However we see from eq. (9) that the vacuum diagram does *not* contribute to $\mathcal{Z}[J]$. This result is only established here up to order $\sim \mathcal{O}(g^2)$, but it is quite generally true, to all orders in g; it is of course assumed in our original discussion of $\mathcal{Z}[J]$ (see eqs. B.1 (22) and B.1 (18)), and follows directly from the original definition in eqs. B.1 (20) and B.1 (17).

The next obvious question that arises here is - what form the correlators $G_n(x_1, \ldots, x_n)$ take in a perturbative expansion? To do this up to $\sim \mathcal{O}(g^2)$, it suffices to take the result we have just derived for $\mathcal{Z}[J]$ in eq. (9). This is straightforward - we find the following results:

A. The 2-point correlator

$$G_2(x_1, x_2) \;=\; -\hbar^2 \frac{\delta^2 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)} \,|_{J=0} \tag{11}$$

(compare eq. B.1 (19)). If we substitute eq. (9) above into this, we get the eqtn. in fig. (12) which we may rewrite, using the usual the relation in eq. B.1 (69) between $G_0(x_1 - x_2)$ and $\Delta_F(x_1 - x_2)$, as

$$G_2(x_1 - x_2) \;=\; G_0(x_1 - x_2) \;-\; \frac{g}{2} G_0(0) \int d^4x' \, G_0(x_1 - x') G_0(x' - x_2) + \mathcal{O}(g^2) \tag{13}$$

with Fourier transfor

$$G_2(k) \;=\; G_0(k) \left[1 - i\frac{g}{2}\Delta - F(0) G_0(k)\right]^{-1} + \mathcal{O}(g^2) \tag{14}$$

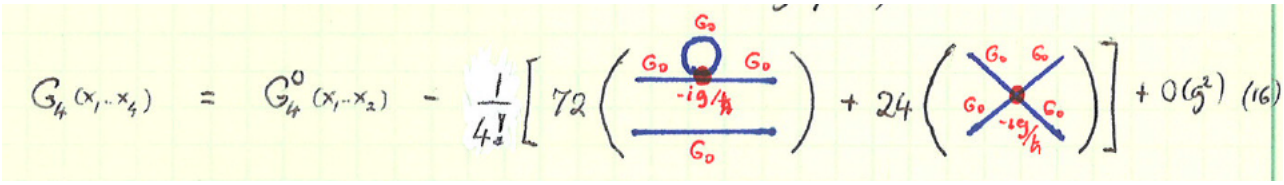
a result which we interpret below.

B. The 4-point correlator

$$G_4(x_1, \ldots, x_4) \;=\; \hbar^4 \frac{\delta^4 \mathcal{Z}[J]}{\delta J(x_1) \cdots \delta J(x_4)} \,|_{J=0} \tag{15}$$

(see eq. B.1 (19)).

The detailed evaluation of this is a tedious but straightforward exercise. The term $\sim \mathcal{O}(g^\circ)$, i.e., independent of g, was already given in eqs. B.1 (70) and B.1 (71). The final



result, after doing the functional differentiations, can be written graphically as the eqtn. in fig. (16) and we see that the 2 diagrams for G_4 that are $\sim \mathcal{O}(g)$ comprise a set of disconnected graphs in which one of the 2 propagators is renormalized, plus a set of connected scattering graphs.

One can derive all the $G_n(x_1, \ldots, x_n)$ for the ϕ^4 theory in a systematic way once we establish a set of rules for the graphs - we do this below.

B.3.1(b) : EXPANSION of $W[J]$ and $\mathcal{G}_n^{(c)}(x_1, \ldots, x_n)$ for ϕ^4 THEORY:

We can tie up all the loose ends here by looking at our other two functionals. Let's consider first $W[J] = -i\hbar\mathcal{Z}[J]$, which, we recall, is supposed to be made up entirely of connected graphs.

We could try working out $W[J]$ directly, but here we are interested in the structure of the correlators, and this we can find out directly from what we have already calculated for $\mathcal{Z}[J]$.

Back in section B.1 we found that there were 2 ways we could compute the correlators $\mathcal{G}_n^{(c)}(x_1, \ldots, x_n)$, i.e., we have

$$\begin{aligned} \mathcal{G}_n^{(c)}(x_1, \ldots, x_n) &= (-i\hbar)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \,|_{J=0} \\ &= (-i\hbar)^n \frac{1}{Z[J]} \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1) \cdots \delta J(x_n)} \,|_{J=0} \end{aligned} \tag{1}$$

and we showed that these were the same, despite appearances. Now it is actually quite illuminating to see how this works out in a diagrammatic expansion, in powers of the coupling g; so let's evaluate the 2-point and 4-point functions $\mathcal{G}_2^{(c)}(x_1, x_2)$ and $\mathcal{G}_n^{(c)}(x_1, \ldots, x_4)$ using both methods.

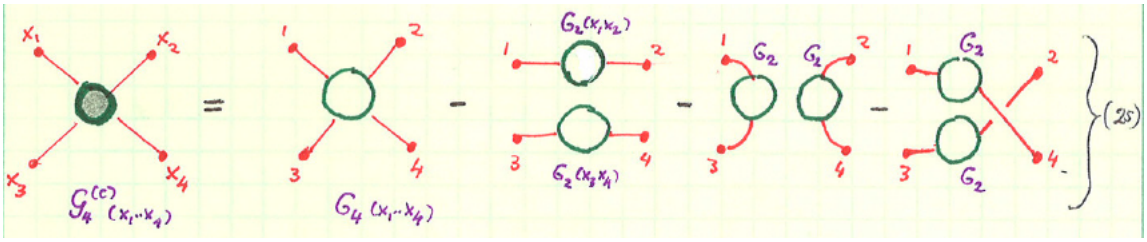
The 2-point correlator $\mathcal{G}_2^{(c)}(x_1, x_2)$ is given in terms of $W[J]$ by

$$\mathcal{G}_2^{(c)}(x_1, x_2) = -i\hbar \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} \,|_{J=0} \tag{18}$$

$$\rightarrow \quad -i\hbar \left\{ -i\hbar \left[\frac{1}{\mathcal{Z}[J]} \frac{\delta^2 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)} - \frac{1}{\mathcal{Z}^2[J]} \frac{\delta \mathcal{Z}}{\delta J(x_1)} \frac{\delta \mathcal{Z}}{\delta J(x_2)} \right] |_{J=0} \right\} \tag{19}$$

where the result (19) comes from using $W = -i\hbar\mathcal{Z}$. Now when $J = 0$, $\delta\mathcal{Z}/\delta J = 0$, and so we get from (19) that

$$\mathcal{G}_2^{(c)}(x_1, x_2) = (-i\hbar)^2 \frac{1}{\mathcal{Z}[J]} \frac{\delta^2 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)} \,|_{J=0} \tag{20}$$



which is exactly what we expect from (17). Moreover, since:

$$\mathcal{Z}[J]\,|_{J=0} \;=\; 1 \tag{21}$$

by definition, we find that

$$\mathcal{G}_2^{(c)}(x_1,x_2) \;=\; G_2(x_1,x_2) \;=\; (-i\hbar)^2\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\,|_{J=0} \tag{22}$$

In other words, we have found that the connected correlator $\mathcal{G}_2^{(c)}(x_1,x_2)$ is equal to the full correlator $G_2(x_1,x_2)$; this has been shown here explicitly without any expansion in powers of g, to all orders in g. If we expand out to linear order in g, we just get eq. (12) again, but now for $\mathcal{G}_2^{(c)}(x_1,x_2)$. This is of course what we would expect - it is impossible to have disconnected graphs for a 2 point function.

The same is not true for 4-point functions. It is then quite interesting to see how things work out in perturbation theory. Let's again begin with $W[J]$. Then from (17) we have

$$\mathcal{G}_4^{(c)}(x_1,\ldots,x_4) \;=\; (-i\hbar)^3\frac{\delta^4 W[J]}{\delta J(x_1)\cdots\delta J(x_4)}\,|_{J=0} \tag{23}$$

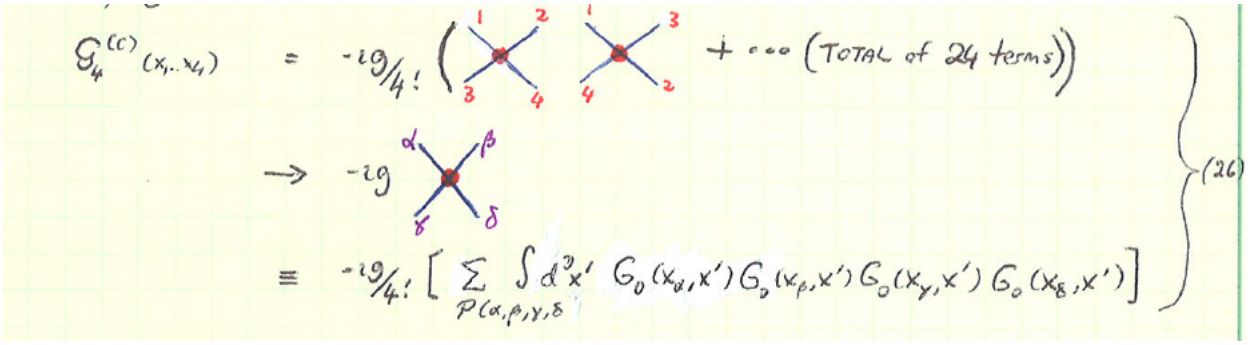
and if we work through the details of this, we easily find that

$$\begin{aligned}\mathcal{G}_4^{(c)}(x_1,\ldots,x_4) \;=\;& G_4(x_1,\ldots,x_4) - G_2(x_1,x_2)G_2(x_3,x_4)\\ &- G_2(x_1,x_3)G_2(x_2,x_4) - G_2(x_1,x_4)G_2(x_2,x_3)\end{aligned} \tag{24}$$

or in the diagrams shown in (25), in which the products of G_2's are given over all possible pairings of x_1,x_2,x_3,x_4.

Now at first glance there seems to be something very wrong with these last 2 equations. On the left-hand side we have a *connected* Green function - i.e., it cannot be disconnected into separate parts - whereas on the right-hand side we have a whole bunch of disconnected terms: both the products of G_2's, and disconnected terms in G_4.

The magic here is that we find, to any order in perturbation theory, that all the disconnected terms on the right-hand side actually cancel. One can actually verify this up to $\sim \mathcal{O}(g^2)$, for the ϕ^4 theory we are considering here, by substituting the expressions (12) and (16) for $G_2(x_1,x_2)$ and $G_4(x_1,\ldots,x_4)$ respectively, into eq. (24) above. The evaluation is a little tedious but entirely straightforward - we eventually get the eqtns. shown in (26),



where in the last expression we sum over all 24 permutations of the external positions x_α, x_β, x_γ, and x_δ.

B.3.1(c) : EXPANSION of $\Gamma[\phi]$ for ϕ^4 THEORY

Finally, let's see what the proper vertices look like in perturbation theory in g. Now we could just go through the same routines as above, but we shall do something a little different here, in anticipation of later developments (to do with the semiclassical and loop expressions). What we are going to do is to see, in perturbation theory, a very interesting relationship between the action $S[\phi]$ and the proper vertex functional $\Gamma[\phi]$.

Consider what happens when we approximate the sum over paths in the expressions we have given up to now, by the classical result, i.e., by the single path of minimum action. This means that if we start with the functional $\mathcal{Z}[J]$, we end up with a function $\bar{Z}_{cl}[J]$;

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\left[S[\phi]+\int d^D x \, J(x)\phi(x)\right]} \tag{2}$$

$$\rightarrow \quad \bar{Z}_{cl}[J] = \exp\left\{\frac{i}{\hbar}\left[\bar{S}_{cl}[\bar{\phi}] + \int d^D x \, \bar{\phi}(x) J(x)\right]\right\} \tag{3}$$

where the "classical" solution $\bar{\phi}(x)$ minimizes the action functional:

$$\frac{\delta S[\phi]}{\delta \phi(x)} \, |_{\phi=\bar{\phi}} + J(x) = 0 \tag{28}$$

and $\bar{S}_{cl} \equiv S[\bar{\phi}]$.

For the ϕ^4 action this just gives

$$(\partial^2 + m^2)\bar{\phi}(x) + \frac{g}{3!}\bar{\phi}^3(x) = J(x) \tag{29}$$

as the classical equation of motion for the $\phi(x)$ field in ϕ^4 theory, in the presence of an external source $J(x)$.

Now we can ask what happens to $W[J] \rightarrow \bar{W}[J]$ and $\Gamma[\bar{\phi}]$ in this classical limit. Let's rewrite eq. (27) as

$$\bar{W}_{cl}[J] = -i\hbar \ln \bar{Z}_{cl}[J] = \bar{S}_{cl}[\bar{\phi}] + \int d^D x \, J(x)\bar{\phi}(x) \tag{30}$$

But this just shows that in this classical limit, the classical action is the same as the generating functional for the proper vertices, i.e.,

$$\bar{\Gamma}_{cl}[\bar{\phi}] \;=\; \bar{S}_{cl}[\bar{\phi}] \qquad \text{(classical limit)} \tag{31}$$

(cf. definition of $\Gamma[\phi]$, eq. B.1 (41)). This result is our first one of the remarkable structure contained in semiclassical expressions in field theory, to which we will return.

Let's see how this works out for the ϕ^4 theory, and solve perturbatively for $\bar{\phi}(x)$ in eq. (29). Since $\Delta_F^{-1}(x) = -(\partial^2 + m^2)$ (ignoring regularizing δ function for now), we have

$$\bar{\phi}(x) \;=\; -\int d^D x'\, \Delta_F(x-x') \left[J(x') - \frac{g}{3!}\bar{\phi}^3(x') \right] \tag{32}$$

where $V'(\phi) = \delta V[\phi]/\delta\phi(x) = \frac{g}{3!}\phi^3(x)$ in ϕ^4 theory. We can now simply find the solution iteratively, by iterating in the parameter $\lambda = \frac{g}{3!}$. We then get

$$\bar{\phi}_{(0)}(x) \;=\; -\int d^D x'\, \Delta_F(x-x') J(x') \tag{33}$$

$$\bar{\phi}_{(n+1)}(x) \;=\; -\int d^D x'\, \Delta_F(x-x') \left[J(x') - \lambda \bar{\phi}^3_{(n+1)}(x) \right] \tag{34}$$

so that, eg.,

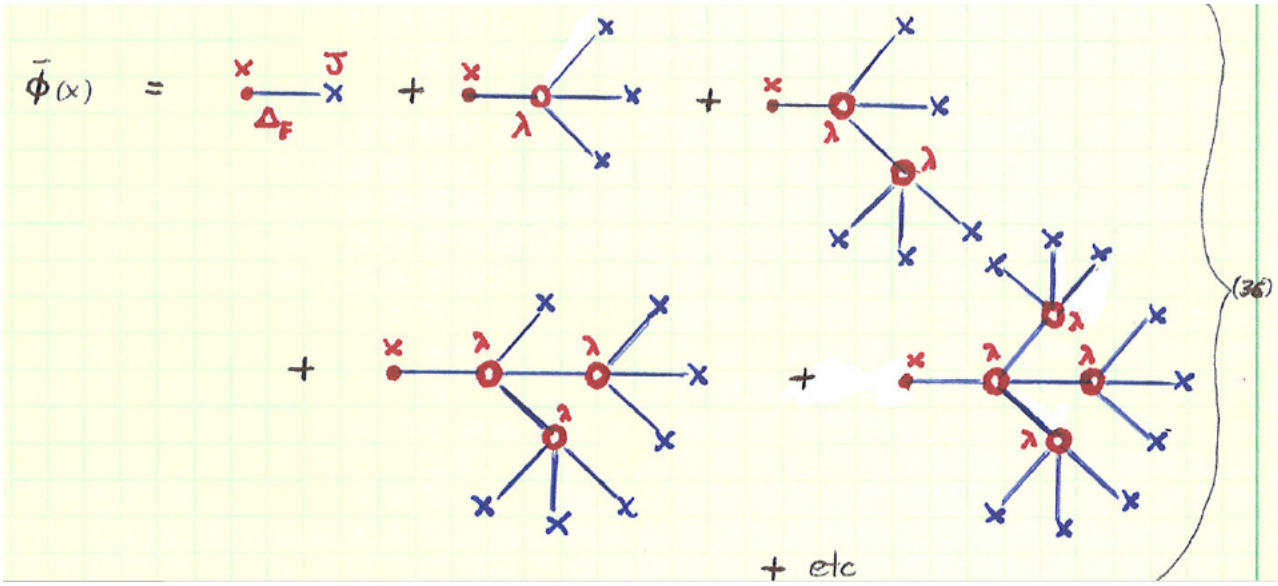
$$\begin{aligned} \bar{\phi}_{(1)}(x) \;&=\; \bar{\phi}_{(0)}(x) \;+\; \lambda \int d^D x'\, \Delta_F(x-x') \bar{\phi}^3_{(0)}(x') \\ &=\; \bar{\phi}_{(0)}(x) \;+\; \lambda \int d^D x'\, \Delta_F(x-x') \left[\int d^D x''\, \Delta_F(x'-x'') J(x'') \right]^3 \end{aligned} \tag{35}$$

If we depict this perturbative expression graphically, we get, for terms up to order λ^3, the terms are like diagrams (36) from which we clearly see the "tree" structure of these graphs for $\bar{\phi}(x)$. From eq. (31) we also have

$$\bar{\Gamma}_{cl}[\bar{\phi}] \;=\; \bar{S}_{cl}[\bar{\phi}] \;=\; \int d^D x\, \bar{\phi}(x) \tag{37}$$

so that in the semiclassical limit, both $\bar{S}_{cl}[\bar{\phi}]$ and $\bar{\Gamma}_{cl}[\bar{\phi}]$ also have this tree structure, with no "loops" (i.e., lines that close back or region other lines). Thus the classical approximation (also referred to commonly as the "tree approximation", is easy to implement graphically (there are actually some subtleties here, which we will come back to in Chapters 6 and 7).

Actually, as we saw from section B.1, all graphs for $\Gamma[\phi]$ have this tree structure, although they do contain "inner loops". We shall look at this more later on, both in the context of diagrammatic expansions, and when we discuss more general results in the context of loop expansions about the classical theory, in the chapter on non-perturbative methods (see Chapter 6).



B.3.2: FEYNMAN RULES - SOME EXAMPLES

In very detailed texts on QFT, in either particle physics or condensed matter physics, the Feynman rules for a given field theory are derived. This can be a lengthy process, and we will not do it here. Another approach, popularized by Hooft and Veltman in 1973 (and of course initially developed by Feynman) is to simply define the perturbative theory by the Feynman rules, and then work out the details from there. With a little intuition and experience it is easy to do this correctly.

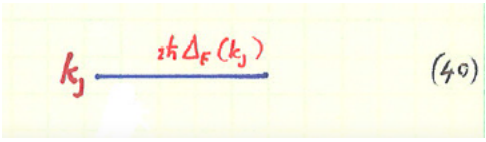
In what follows I will do this for a few simple field theories, of central importance in particle physics and condensed matter physics. One thing I will not do here, but is done later. So here we will simply look at self-interesting fields, by way a gentle introduction. We will also ignore any order parameter condensation.

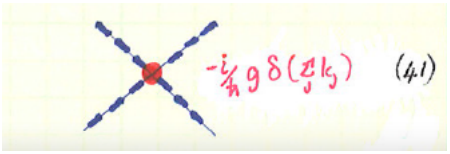
B.3.2(a) : FEYNMAN RULES for ϕ^4 THEORY

From the calculations of graphs that we have already done, it is pretty easy to guess the Feynman rules for this theory; and it is also easy to see how they come about. The key is to start from the defining formulas

$$\begin{aligned} \mathcal{Z}[J] &= \mathcal{N}\, e^{\frac{i}{\hbar}\int d^D x\, V\left[-i\hbar\frac{\delta}{\delta J(x)}\right]}\, \mathcal{Z}_0[J] \\ \mathcal{Z}_0[J] &= e^{\frac{-i}{2\hbar}\int d^D x \int d^D x' J(x)\Delta_F(x-x')J(x')} \end{aligned} \tag{38}$$

which we repeat here for good measure - cf. eqs. (79) and (80) in section B.1. The normalization factor $\mathcal{N}$ is just $\mathcal{Z}[J=0]$, and we will not need it here. We can also write this





as

$$\mathcal{Z}[J] \;=\; \mathcal{N}\mathcal{Z}_0\left[-i\hbar\frac{\delta}{\delta\phi(x)}\right]\; e^{\frac{-i}{\hbar}\int d^Dx\,[V(\phi)-\phi(x)J(x)]}\;|_{\phi=0}$$
$$\rightarrow\;\; \mathcal{N}\, e^{\frac{-i}{2\hbar}\int d^Dx\int d^Dx'\,\frac{\delta}{\delta\phi(x)}\Delta_F(x-x')\frac{\delta}{\delta\phi(x')}}\;\; e^{\frac{-i}{\hbar}\int d^Dx\left[\frac{g}{4!}\phi^4(x)-J(x)\phi(x)\right]}\;|_{\phi=0} \qquad (39)$$

involving functional derivatives $\phi(x)$; cf. eqs. (83) and (84) of section B.1. Either (38) or (39) can be used to generate diagrammatic expressions. Let us first give the Feynman rules for this theory, and then see how they emerge from either (38) or (39). The rules can be expressed as follows:

For some quantity having n external legs (eg., the correlator $G_n(x_1,\ldots,x_n)$, or the vertex part $\Gamma_n(x_1,\ldots,x_n)$), we draw all possible graphs having n external legs, and an arbitrary number of internal interaction vertices. Graphs for $g_n^{(c)}(x_1,\ldots,x_n)$ and $\Gamma_n(x_1,\ldots,x_n)$ are connected, whereas diagrams for $G_n(x_1,\ldots,x_n)$ do not have to be.

Then, to the graphs, we assign values according to:

1. Each line has the value $i\hbar\Delta_F(k_j)$ (and momentum k_j) in the diagram (40), where k_j is the momentum of the line.

2. Each interaction vertex in the ϕ^4 theory, with interaction term $-\frac{g}{4!}\phi^4(x)$ in the Lagrangian, contributes the factor $(-i/\hbar)g\,\delta(\sum_j k_j)$ in the diagram (41), where we sum over the momenta k_j which flow into or out of the vertex (the δ function expresses momentum conservation).

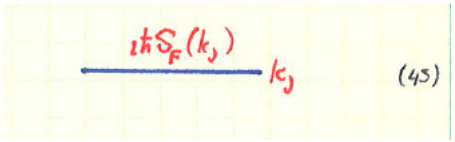
3. All independent momenta in the diagram, not fixed by momentum conservation, are integrated over, i.e., we multiply by the factor

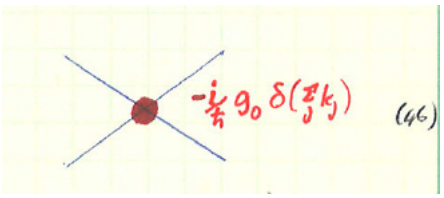
$$\prod_{\alpha=1}^{L}\sum_{k_\alpha} \;=\; \int\frac{d^Dk_1}{(2\pi)^D}\int\frac{d^Dk_2}{(2\pi)^D}\cdots\int\frac{d^Dk_L}{(2\pi)^D} \qquad (42)$$

The number L of independent internal momenta (the number of "Loops") is given in terms of the number I of internal lines, and the number V of internal vertices, is given by

$$L \;=\; I+1-V \qquad (43)$$

4. Each diagram is multiplied by a SYMMETRY FACTOR, or combinatorial factor. This is defined by the total number of different ways of generating the same diagram.





The definition and calculation of symmetry factors doesn't pose any difficulties of principle, but is a little finicky - we discuss the procedure in an appendix.

We have already worked out quite a few diagrams for the ϕ^4 theory in the proceeding pages, so there is no need to give any examples here. Note that although the expressions given above can be derived easily from the Feynman rules given above, we still have not actually evaluated them, i.e., we have not done the 4-momentum integrals.

B.3.2(b) : FEYNMAN RULES for DIRAC FERMIONS

We have already discussed the case of non-interacting Dirac fermions - but what if they are interacting with each other? There are actually a number of different ways this can happen, but in this course we will concern ourselves with 2 kinds of self-interaction, i.e.,

(i) A short-range relativistic point interaction;

(ii) A non-relativistic electron-electron interaction arising from the Coulomb interaction in condensed matter systems.

We deal here with the 1st case; the second case is discussed immediately below. This first case will be defined by the local interaction

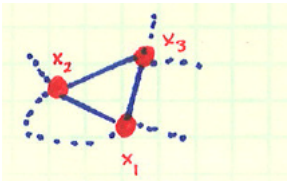
$$L_{int} \quad = \quad -g_0 \int d^4x \, (\bar{\psi}(x)\psi(x))^2 \tag{44}$$

Then the rules are given as follows:

1. Each fermion line has the propagator $G_0(k_j) = i\hbar S_F(k_j)$ associated with it in the diagram (45), where k_j is the 4-momentum of the line.

2. Each interaction vertex, coming from the interaction (44), contributes the factor $(-i/\hbar)g\,\delta(\sum_j k_j)$ in the diagram (46), where we sum over the momenta flowing into or out of the vertex.

3. All independent momenta in the diagram are integrated over; this is like the bosonic ϕ^4 theory except that there is an extra factor $(-1)^L$, where l is the number of fermion loops;



thus we have to integrate according to

$$(-1)^L \prod_{\alpha=1}^{L} \sum_{k_\alpha} \;=\; (-1)^L \int \frac{d^D\, k_1}{(2\pi)^D} \cdots \int \frac{d^D\, k_L}{(2\pi)^D} \tag{47}$$

4. Again, each diagram, is multiplied by a symmetry factor.

The fermion loop factor $(-1)^L$ comes from the antisymmetry under the exchange of fermion fields. For some loop, we will be presented with a sequence of fields of form

$$\hat{T}\, \{ \cdots \psi(x_1) \bar{\psi}(x_2) \psi(x_2) \cdots \bar{\psi}(x_L) \psi(x_L) \bar{\psi}(x_1) \cdots \} \tag{48}$$

We can describe (48) by the diagram, where the dotted lines in the graph at the top, mediate lines not involved in the loop. But to order this sequence, we must move the last field $\bar{\psi}(x_1)$ to the beginning of the sequence - and this will give a sign of -1, no matter what is L or what are the arguments of the vertices (you might, in order to make sure of this argument, change everything to the momentum representation, and/or work this out for an example - eg., a simple graph with one or two loops).

B.3.2(c) : FEYNMAN RULES for NON – RELATIVISTIC FERMIONS

This last case is of great importance in condensed matter physics (and in astrophysics). This is because of the prevalence in Nature of finite density systems of mobile fermions (in metals, superconductors, white dwarfs, neutron stars, etc.).

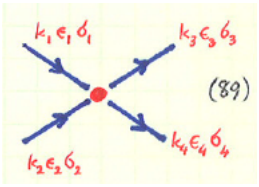
We already saw in the last section what the action for a set of mobile fermions has - recall that for a finite density of fermions, the Fermi surface $S_F(\mathbf{k})$ is defined by the chemical potential, so that for a non-interacting Fermi gas, we have the result that

$$\mathbf{k} \;\in\; S_F(\mathbf{k}) : \quad \text{iff} \quad \epsilon^o_{\mathbf{k}} = \mu \tag{84}$$

Later we will see that in the presence of interactions, the Fermi surface $S_F(\mathbf{k})$ can move in $\mathbf{k}$-space, as a function of the interactions.

We have already seen the form of the generating functional as well (Chapter B2, eqs. (60)-(62)); we have

$$\mathcal{Z}[\bar{\eta}, \eta] \;=\; \int \mathcal{D}\bar{\psi}_{\mathbf{k}\sigma} \int \mathcal{D}\psi_{\mathbf{k}\sigma}\; e^{\frac{i}{\hbar}\left[S[\bar{\psi},\psi] + \int d^4x\, (\bar{\eta}_{\mathbf{k}\sigma}\psi_{\mathbf{k}\sigma} + \bar{\psi}_{\mathbf{k}\sigma}\eta_{\mathbf{k}\sigma})\right]} \tag{85}$$



where we are assuming here an isotropic, translationally invariant system, so that $\mathbf{k}$ is a good quantum number - examples include bulk ^3He, bulk nuclear matter, or a cold Fermi gas, or a star. In a conducting solid, neither $\mathbf{k}$ nor σ are good quantum numbers (because of the lattice, and because of spin-orbit coupling); and in the nuclear, which is finite, we must classify states according to the symmetry group of the nucleus, as in the atom.

For the homogeneous system, in the presence of interactions, we have an action

$$S = S_0[\bar{\psi},\psi] + S_{\rm int}[\bar{\psi},\psi] \tag{86}$$

from B.2, eqs. (60) and (64), we have

$$\begin{aligned} S_0[\bar{\psi},\psi] &= \int \frac{d\epsilon}{2\pi} \sum_{\mathbf{k}\sigma} \bar{\psi}_{\mathbf{k}\sigma}(\epsilon) \left[\hbar\epsilon - \epsilon^0_{\mathbf{k}} + \mu\right] \psi_{\mathbf{k}\sigma}(\epsilon) \\ S_{\rm int}[\bar{\psi},\psi] &= -\int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi} \sum_{\mathbf{k},\mathbf{q}} \sum_{\sigma,\sigma'} V(\mathbf{q},\omega) \bar{\psi}_{\mathbf{k}\sigma}(\epsilon)\psi_{\mathbf{k}\sigma}(\epsilon)\bar{\psi}_{\mathbf{k}+\mathbf{q},\sigma'}(\epsilon+\omega)\psi_{\mathbf{k}+\mathbf{q},\sigma'}(\epsilon+\omega) \end{aligned} \tag{87}$$

where we have generalized the interaction $V(\mathbf{q})$ in B.2 eq. (60), to include a frequency dependence as well. Then the Feynman rules for this system are:

1. Each fermion line, at $T = 0$, is assigned the value

$$G_0^{\sigma\sigma'}(\mathbf{k},\epsilon) = i\hbar\delta_{\sigma\sigma'} \left[\frac{\theta(|\mathbf{k}| - k_F)}{\hbar\epsilon - (\epsilon^0_{\mathbf{k}} - \mu) + i\delta} + \frac{\theta(k_F - |\mathbf{k}|)}{\hbar\epsilon - (\epsilon^0_{\mathbf{k}} - \mu) - i\delta} \right] \tag{88}$$

where we separate the "particle" and "hole" excitations (existing above and below the Fermi surface respectively). The proof that G takes this form will be given later, when we deal with temperature Green functions. Note that if spin is not conserved (e.g., if we have spin-orbit coupling) then the $\delta_{\sigma\sigma'}$ is suppressed. We can think of (85) as the sum of particle and hole terms, with the propagator direction depending on which we deal with - note the opposite signs of the δ-functions.

2. The interaction vertex for the action in (87) is given by

$$-\frac{i}{\hbar} V(\mathbf{q},\omega)\ \delta\Big(\sum_j \mathbf{k}_j\Big)\ \delta\Big(\sum_j \epsilon_j\Big)\ \delta\Big(\sum_j \sigma_j\Big)$$

where we enforce the conservation of momentum, energy, and spin at each vertex, as shown in the diagram (89).

3. All independent momenta, frequencies, and spin, projections are summed over; and a factor $(-1)^L$ appears, where L is the number of independent loops in the diagram.

4. A symmetry factor is associated with each diagram.

This concludes our introductionary discussion of Feynman rules - anyone wanting much more detail and a deeper understanding should go to books, or to the "Diagrammar" article of 't Hooft and Veltman. Note again that so far we have only dealt with self-interacting fields, and that the symmetry factors have not been specified. In section B.3.4 we will remove the first of these lacunae, by discussing systems where two different fields couple to each other.

B.3.3: PHONONS in SOLIDS and LIQUIDS

When we turn to non-relativistic systems, things rapidly get more complex, for several reasons. These include

(i) the existence in solids of a lattice, either ordered or disordered - we can no longer simplify things by going to a momentum basis;

(ii) the finite density of excitations in the system, or of particles - this is particularly important for fermions, as it creates a fermi sea;

(iii) the importance of temperature - we can no longer assume a background ground state.

And finally, of course, many condensed matter systems have order parameters, created either by a spontaneously broken symmetry, or otherwise. The existence of this order parameter fundamentally changes the physics, and also changes the Feynman rules (and this is also true in high-energy physics, in the standard model, where we also have spontaneously broken symmetries).

We will begin simply here by looking at the case of ordinary phonons, which are the quantized version of the small-amplitude oscillations that occur in any solid, liquid, or gaseous medium. In a solid these are sometimes called "lattice vibrations", although they occur whether or not the "lattice" is ordered, and do not just involve the lattice ions. In a quantum liquid at low T phonons are also well-defined, both in systems like Bose-condensed liquids, and also in Fermi liquids; and they are also well-defined in low-T gases. In all of these systems they lose their definition when their density and/or the temperature is high (except at long wavelengths), because of strong phonon-phonon interactions, or because the compressional or shear oscillations of the system are driven by collisional pressure processes.

In what follows here we will discuss phonons in an introductory way. Thus we will derive the Feynman rules in the long wavelength regime, and discuss phonons in simple solids, either ordered ones of simple symmetry, or disordered ones, and also in quantum liquids. We will also ignore the interaction with electrons, impurities, or phase oscillations (in a superfluid). The resulting theory then looks rather similar to the ϕ^4 model of relativistic field theory, in the limit where the mass $m \to 0$.

The key difference between solids and liquids, as far as oscillations of the medium are concerned, is that solids have a finite resistance to shear forces, and so can support transverse

shear oscillations. A liquid, by definition, will simply flow under shear forces, and so can only support longitudinal compressional modes.

B.3.3(a) : PHONONS in LIQUIDS

In this case we assume the fundamental conservation equation, describing mass conservation

$$\frac{d}{dt}\rho(\mathbf{r},t) \;=\; \partial_t\rho + \nabla(\rho\mathbf{v}) = 0 \tag{49}$$

where $\rho(\mathbf{r},t)$ is the local density, and $\mathbf{v}(\mathbf{r},t)$ the local velocity of the fluid. So far this is a classical description. One can then proceed in one of 2 ways.

One way is to simply treat the density and velocity as canonical variables, and quantize them with the appropriate commutation relations, between density and velocity operators. The other way is to start from an action for the fluid, and then use this as input for a path integral formulation. When we come to look at phonon properly we will look at both, but here let's just start with the action, which we will write in terms of the variables

$$\begin{aligned}\rho(\mathbf{r},t) &= \rho_0 + \tilde{\rho}(\mathbf{r},t) \qquad (\rho_0 = \text{mean density})\\ \mathbf{v}(\mathbf{r},t) &= (\hbar/m)\nabla\phi(\mathbf{r},t)\end{aligned} \tag{50}$$

where we introduce the "density fluctuation" variable $\tilde{\rho}(\mathbf{r},t)$, and assume that $\tilde{\rho}/\rho_0 \ll 1$ (appropriate to liquids); and the "quantum phase" variable ϕ (we could equally have ignored QM altogether, and written $\mathbf{v}(\mathbf{r},t) = (1/m)\nabla\Phi(\mathbf{r},t)$, with $\Phi = \hbar\phi$; this is purely a question of definition). Then if we write the action as

$$S \;=\; -\int d^D\mathbf{r}\int dt\,\left\{\frac{\rho}{m}\left[\hbar\frac{d\phi}{dt} + \frac{\hbar^2}{2m}(\nabla\phi)^2\right] + \epsilon[\rho,\nabla\rho]\right\} \tag{51}$$

we find, by varying S with respect to ϕ and $\tilde{\rho}$, the following equations of motion:

$$\begin{aligned}&\text{(a)}\quad \partial_t\rho(\mathbf{r},t) + \frac{\hbar}{m}\nabla\cdot[\rho(\mathbf{r},t)\nabla\phi(\mathbf{r},t)] \;=\; 0 \qquad\qquad \text{(Mass Conservation)}\\ &\text{(b)}\quad \hbar\partial_t\phi(\mathbf{r},t) + \frac{\hbar^2}{2m}\left[\nabla\phi(\mathbf{r},t)\right]^2 + \frac{\delta\epsilon[\rho,\nabla\rho]}{\delta\rho(\mathbf{r},t)} \;=\; 0 \qquad\qquad \text{(F=ma)}\end{aligned} \tag{52}$$

where the last term in eq. (52.b) is the variation with density of the internal energy density $\epsilon[\rho,\nabla\rho]$ of the fluid, as a functional of ρ and $\nabla\rho$. Eq. (52.a) is easily seen to be equivalent to the mass conservation equation in (49). If we write the internal pressure $P(\mathbf{r},t)$ of the liquid as $P(\mathbf{r},t) = \rho\delta\epsilon/\delta\rho(\mathbf{r},t)$, and then take the gradient of eq. (52.b), we get

$$\frac{d}{dt}\mathbf{v}(\mathbf{r},t) \;=\; \partial_t\mathbf{v}(\mathbf{r},t) + (\mathbf{v}\cdot\nabla)\mathbf{v}(\mathbf{r},t) \;=\; -\frac{1}{\rho}\nabla P(\mathbf{r},t) \tag{53}$$

which is just the Euler equation of motion for a fluid.

To get the phonons, we now look at small oscillations of the action about its minimum. In a relativistic field theory, this would be akin to treating the underlying fluid, with action S_{min}, as the "vacuum", and the small oscillations are then the field we actually look at explicitly. The result of a 2nd variation of S in eq. (51) is

$$\begin{aligned} S &= S_0 - \int d^D\mathbf{r} \int dt \left[\frac{\hbar}{m}\tilde{\rho}\partial_t\phi + \frac{\hbar^2\rho}{2m^2}(\nabla\phi)^2 + \frac{1}{2}c_0^2\tilde{\rho}^2\right] \\ &\sim S_0 - \int d^D\mathbf{r} \int dt \left[\frac{\hbar}{m}\tilde{\rho}\partial_t\phi + \frac{1}{2}c_0^2\tilde{\rho}^2\right] \end{aligned} \tag{54}$$

where in the 2nd form we drop the term $\propto v^2$, assuming we are dealing with small oscillations of long wavelength; here we define $c_0^2 = \partial P/\partial\rho$.

Now it is straightforward to show, starting from either (54) or from (52), that the small oscillations in this system are actually sound waves. Suppose we write (52), now dropping the $(\nabla\phi)^2$ term so was done in (54), in the form

$$\begin{aligned} \partial_t\rho + \nabla\cdot\mathbf{J} &= 0 \\ \partial_t\mathbf{J} + \nabla\rho &= 0 \end{aligned} \tag{55}$$

where $\mathbf{J}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)$.

We may now combine these 2 equations, by taking the time derivative of the first equation, to get

$$\big(\partial_t^2 - c_0^2\nabla^2\big)\rho(\mathbf{r},t) = 0 \tag{56}$$

and we see that c_0 is the sound velocity.

B.3.3 (b): PHONONS in SOLIDS

The case of solids is somewhat different because we can start with an underlying set of lattice points (ordered or not), and write everything in terms of the displacement $\mathbf{x}_j = \mathbf{r}_j - \mathbf{r}_j^0$ of the j-th lattice point from its equilibrium position $\mathbf{r}_j^0$. Alternatively, in a continuum approximation, one simply defines a displacement vector $\mathbf{x}(\mathbf{r})$ as a function of $\mathbf{r}$ in the system. Then the following results are a standard part of elasticity theory:

(i) The change in free energy F (and at $T = 0$, the system energy) associated with a determination of a solid is given by

$$F - F_0 = \frac{1}{2}\sum c_{ijkl}u^{ij}(\mathbf{r})u^{kl}(\mathbf{r}) + \cdots \tag{57}$$

where the $u_{ij}(\mathbf{r})$ are the components of the strain tensor, defined as

$$u_{ij}(\mathbf{r}) = \frac{1}{2}\big(\partial_i\mathbf{x}_j + \partial_j\mathbf{x}_i\big) = \frac{1}{2}\big(\frac{\partial\mathbf{x}_i}{\partial\mathbf{r}^j} + \frac{\partial\mathbf{x}_j}{\partial\mathbf{r}^i}\big) \tag{58}$$

(ii) If a force $\mathbf{F}(\mathbf{r})$ is applied to the solid in order to deform it - and here we assume that $\mathbf{F}(\mathbf{r})$ can vary around the sample - then the change in energy is

$$\delta E \;=\; \sigma_{ij}(\mathbf{r})\delta u^{ij}(\mathbf{r}) \tag{59}$$

where the force $\mathbf{F}$ is related to the stress tensor $\sigma_{ij}(\mathbf{r})$ by

$$F_j \;=\; \partial^k \sigma_{jk} \;=\; \frac{\partial \sigma_{jk}}{\partial r_k} \tag{60}$$

Now for a general solid the coefficients $c_{ijkl}(\mathbf{r})$ may be very complicated; even for a homogeneous system which is crystalline, one needs to employ group theory to sort all this out. We will typically look at a homogeneous and isotropic system (e.g., and amorphous solid, or one made from a mass of tiny crystals, organized into a random conglomerate or "polycrystalline array", or a very highly disordered crystal). Under these circumstances equation (57) simplifies considerably; we can write

$$\begin{aligned} F \;&=\; F_0 + \frac{1}{2}\left[\lambda u_{kk}^2 + 2\mu u_{ij}^2\right] \\ &=\; F_0 + \frac{1}{2}\left[\kappa u_{kk}^2 + \mu(u_{ij} - \frac{1}{3}\delta_{ij}u_{kk}^2)\right] \end{aligned} \tag{61}$$

where

$$\kappa \;=\; \lambda + \frac{2}{3}\mu \tag{62}$$

and we have taken advantage in the 2nd form in (61) of the separation of u_{ij} into its traceless and diagonal parts, i.e.,

$$\begin{array}{ccccc} u_{ij} & = & \frac{1}{3}\delta_{ij}u_{kk} & + & (u_{ij} - \frac{1}{3}\delta_{ij}u_{kk}) \\ & & \text{(compression)} & & \text{(shear)} \end{array} \tag{63}$$

in which the traceless part describes a pure shear, and the diagonal part a pure compression expression. The coefficients λ and μ are the "Lame" coefficients; the coefficients κ and μ are usually referred to as the "Bulk modulus" and "Shear modulus" respectively. The bulk modulus is simply the inverse of the compressibility:

$$\kappa^{-1} \;=\; -\frac{1}{V}\frac{\partial V}{\partial P} \tag{64}$$

We can eliminate σ_{ij} from (59) for this isotropic case, and easily find for the isotropic case that

$$\sigma_{ij} \;=\; \kappa\,\delta_{ij}u_{kk} \;+\; 2\mu\left(u_{ij} - \frac{1}{3}\delta_{ij}u_{kk}\right)^2 \tag{65}$$

thereby separating σ_{ij} into its compression and shear components respectively.

We note that for crystals, this becomes external messy - then c_{ijkl} can have up to 21 different components independent from each other. If the crystal is magnetic, with spin-orbit coupling, it gets far worse again.

Consider now the form that oscillations will take (which will lead us to the form of the propagator). This is easily found by composing (57) and (59), and taking the strain to come from an applied "source" force given by (60). Then we easily find that

$$F_i \;\; = \;\; \sum c_{ijkl}\, \partial^j u^{kl} \tag{66}$$

which for the simple isotropic case gives, after Fourier transformation to momentum space of (66), the result (here $\hat{q}_i = q_i/|\mathbf{q}|$):

$$x_i(\mathbf{q}) \;\; = \;\; \frac{1}{q^2}\left[\mu(\delta_{ij} + \frac{1}{3}\hat{q}_i\hat{q}_j) + \kappa\hat{q}_i\hat{q}_j\right]^{-1} F^j(\mathbf{q}) \tag{67}$$

To determine the dynamics, we need to add an inertial term to the energies, i.e., use Newton's 2nd law,

$$F_i = \rho\ddot{x}_i \tag{68}$$

where as before, $\rho(\mathbf{r}, t)$ is the density, which we assume constant. Fourier transforming, we then have

$$\rho\omega^2 x_i(\mathbf{q},\omega) - q^2\left[\mu P^{\perp}_{ij}(\hat{q}) + (\kappa + 4\mu/3)P^{\parallel}_{ij}(\hat{q})\right] x^j(\mathbf{q},\omega) \;\; = \;\; 0 \tag{69}$$

where we have defined

$$P^{\parallel}_{ij}(\hat{q}) = \hat{q}_i\hat{q}_j \;; \qquad\qquad P^{\perp}_{ij}(\hat{q}) = (\delta_{ij} - \hat{q}_i\hat{q}_j) \tag{70}$$

i.e., projection operators onto longitudinal and transverse components of $\hat{\mathbf{q}}$, respectively. We then see that we have two sound modes, satisfying

$$(\omega^2 - c^2_\alpha q^2)x^\alpha(\mathbf{q},\omega) \;\; = \;\; 0 \tag{71}$$

with $\alpha = \perp, \parallel$, and

$$\begin{aligned} c^2_{\parallel} &= (\kappa + 4\mu/3)/\rho \\ c^2_{\perp} &= \mu/\rho \end{aligned} \tag{72}$$

as the 2 sound velocities. The transverse sound, with velocity $c_\perp$, is a shear mode, absent in liquids.

B.3.3 (c): FEYNMAN RULES for PHONONS

From what we have done above, it is now a fairly simple matter to get the Feynman rules. Let us first write down the generating functional for these 2 theories. For the liquid we simply go back to (54); however there is a complicating factor, since it is written in

terms of a "coordinate" $\tilde{\rho}(\mathbf{r},t)$ and a "momentum" $\partial_t\phi(\mathbf{r},t)$, and so we will write the bare generating functional as

$$\mathcal{Z}_0[J] \;=\; \int \mathcal{D}\phi \int \mathcal{D}\tilde{\rho}\; e^{\frac{-i}{\hbar}\int d^D\mathbf{r}\int dt\left\{\left[\frac{\hbar}{m}\tilde{\rho}(\mathbf{r},t)\partial_t\phi(\mathbf{r},t)+\frac{1}{2}c_0^2\tilde{\rho}^2\right]-J(\mathbf{r},t)\tilde{\rho}(\mathbf{r},t)\right\}} \tag{73}$$

where we have coupled the external source to the density fluctuations $\tilde{\rho}(\mathbf{r},t)$, which is the natural choice for the coordinate. (although in discussing superfluids, it proves to be useful to couple to the phase).

Rather than integrate out the phase, we will use a simpler method to find the diagram rules. Knowing that the propagator $\mathcal{G}_2^{\tilde{\rho}\rho}(\mathbf{q},\omega)$ is found by calculating a correlator $\langle 0 \mid T\left\{\tilde{\rho}(\mathbf{r},t)\rho(\mathbf{r}',t')\right\} \mid 0\rangle$ (and Fourier transforming it) we will simply calculate this correlator.

To do this let's write the fields $\phi(\mathbf{r},t)$ and $\tilde{\rho}(\mathbf{r},t)$ as

$$\begin{aligned}
\phi(\mathbf{r},t) &= \sum_{\mathbf{q}} \left(\frac{\hbar c_0}{2\rho_0\mathbf{q}}\right)^{\frac{1}{2}} \left[b_{\mathbf{q}}\, e^{i(\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t)} + b_{\mathbf{q}}^\dagger\, e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t)}\right] \\
\tilde{\rho}(\mathbf{r},t) &= i\sum_{\mathbf{q}} \left(\frac{\hbar\rho_0\mathbf{q}}{2c_0}\right)^{\frac{1}{2}} \left[b_{\mathbf{q}}\, e^{i(\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t)} - b_{\mathbf{q}}^\dagger\, e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_{\mathbf{q}}t)}\right]
\end{aligned} \tag{74}$$

where we have normalized things so that

$$[b_{\mathbf{q}},\, b_{\mathbf{q}'}^\dagger] = \delta_{\mathbf{q}\,\mathbf{q}'} \tag{75}$$

Then the Hamiltonian becomes (treated just as an energy-no operators):

$$H \;=\; \frac{1}{2}\left[\frac{\rho}{m}v^2 + c_0^2\tilde{\rho}^2\right] \;=\; \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}}\left[b_{\mathbf{q}}^\dagger b_{\mathbf{q}} + \frac{1}{2}\right] \tag{76}$$

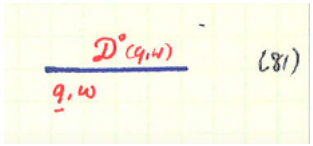
and then we easily find that $\mathcal{D}^0(\mathbf{r},\mathbf{r}';t,t') = \langle 0 \mid T\left\{\tilde{\rho}(\mathbf{r},t)\tilde{\rho}(\mathbf{r}',t')\right\} \mid 0\rangle$ works out to be

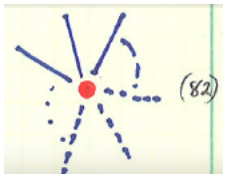
$$\mathcal{D}^0_{\rho\rho}(\mathbf{r}-\mathbf{r}',t-t') = -i\hbar\sum_{\mathbf{q}} \frac{\rho_0\mathbf{q}}{c_0}\left\{\theta(t-t')e^{i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_{\mathbf{q}}(t-t')]} + \theta(t'-t)e^{i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')+\omega_{\mathbf{q}}(t-t')]}\right\} \tag{77}$$

where $\omega_{\mathbf{q}} = c_0\mathbf{q}$ (the "-" sign here compared to that in the function $G_2^{(0)}(\mathbf{x},\mathbf{x}')$ defined for the ϕ^4 theory comes because of the i factor in the definition of $\tilde{\rho}(\mathbf{r},t)$ in (75)). The Fourier transform of this is

$$\begin{aligned}
\mathcal{D}^0_{\rho\rho}(\mathbf{q},\omega) &= \frac{\rho_0}{2c_0}\hbar\mathbf{q}\left[\frac{1}{\omega-(\omega_{\mathbf{q}}-i\delta)} - \frac{1}{\omega+(\omega_{\mathbf{q}}-i\delta)}\right] \\
&= \hbar\rho_0\Big(\frac{q^2}{\omega^2-c_0^2q^2+i\delta}\Big)
\end{aligned} \tag{78}$$

where the subscript "$\rho\rho$" is for $\langle\rho\rho\rangle$. (We could also, if we had wanted, calculated $\mathcal{D}^0_{\phi\phi}(\mathbf{r},t) = \langle 0 \mid T\left\{\phi(\mathbf{r},t)\phi(0,0)\right\} \mid 0\rangle$ directly.)





The case of the isotropic solid is a little different, but the basic idea is the same. We now have a generating functional which in the general case can be written as

$$\mathcal{Z}_0[J_{ij}] \;=\; \int \mathcal{D}x_k \int \mathcal{D}u_{ij} e^{\frac{i}{\hbar}\int d^D\mathbf{r}\int dt\,\left\{\frac{1}{2}\left[\rho_0(\partial_t x_k)^2 - c_{ijkl}u_{ij}u_{kl}\right] + u_{ij}J_{ij}\right\}} \tag{79}$$

where $J_{ij}(\mathbf{r},t)$ is an external source, having the nature of an external stress. If we had instead decided to couple to an external vector J_k, coupled to x_k, this would have been instead an external force source. We could also rewrite the u_{ij} in terms of the internal stress tensor $\sigma_{ij}(\mathbf{r},t)$, and then functionally integrate over σ_{ij}, instead of over $u_{ij}(\mathbf{r},t)$.

Without going through the details, it will now be clear that the bare propagator for the phonons in the isotropic solid will be given by

$$\mathcal{D}^0_{ij}(\mathbf{q},\omega) = \hbar\rho_0 \sum_{\alpha=\parallel,\perp} \hat{P}^\alpha_{ij}(\hat{\mathbf{q}})\Big(\frac{q^2}{\omega^2 - c_\alpha^2 q^2 + i\delta}\Big) \tag{80}$$

i.e., the sum of a longitudinal and transverse term, each travelling with different velocity.

We can now enunciate the Feynman rules for phonons:

1. As shown in the diagram (81), each phonon line has the value $\mathcal{D}^0(\mathbf{q},\omega)$, with $\mathcal{D}^0_{ij}(\mathbf{q},\omega)$ given by (78) for a liquid, and by (80) for an isotropic solids; here $\mathbf{q}$ is the phonon spatial momentum, and ω the frequency.

2. There will be a set of interaction vertices in the phonon system in the diagram (82) - the general interaction taken the form, for the isotropic systems:

$$\text{(a) Liquids:} \qquad V_{\rm int}(\phi,\tilde{\rho}) \;=\; \sum_{n=3}^{\infty} \frac{g_{nm}}{n!}\tilde{\rho}^m(\nabla\phi)^{n-m}$$

$$\text{(b) Solids:} \qquad V_{\rm int}(\partial_t x_k,\, \pi_{ij}) \;=\; \sum_{n=3}^{\infty} \frac{g_{nm}}{n!}(\partial_t x_k)^m(\pi_{ij})^{n-m}$$

These interactions are typically very complicated, even for these isotropic systems (for lattice solids it is much worse). As a consequence one has factors

$$-\frac{i}{\hbar}g_{n,m}\,\delta\Big(\sum_j \mathbf{q}_j\Big)\,\delta\Big(\sum_j \omega_j\Big) \tag{83}$$

for each vertex (in a solid, the momentum delta function is replaced by $\delta\big(\sum_j \mathbf{q}_j + \mathbf{G}_j\big)$, where the $\{\mathbf{G}_j\}$ are reciprocal lattice vectors).

3. All independent momenta and frequencies are integrated over.

4. Each diagram is multiplied by a symmetry factor.

The problem of interacting phonons is seen to be very messy, since one can have interactions of arbitrary order. They are physically important, but not very enlightening, at least for a theorist.

B.3.4: COUPLED SCALAR FIELDS

There is a multiplicity of fields in Nature. Even though physicists have succeeded in encapsulating a large number of these into the "standard model" (with the notable exception of gravity), we also need to consider all the interactions between the different fields. In condensed matter physics, one also needs to consider effective theories in different energy ranges, where the relevant fields, and the interactions between them, change with energy scale.

Thus it is crucial to set up QFT's for interacting fields, and by far the most important example of this is the interaction between fermion matter fields and the bosonic fields which then act to mediate the indirect coupling between the fermions. The most important example of such fermionic-bosonic interactions arises in gauge theories, which we deal with in the next chapter. But there are also key examples that do not involve gauge fields.

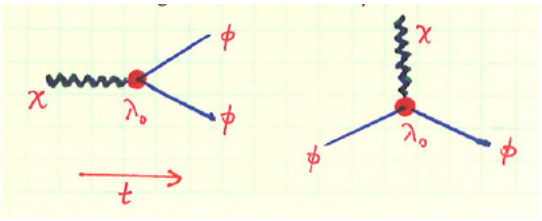
In what follows we will do two things. First, we look at a very simple relativistic model of 2 coupled scalar fields, and then we see how this translates into diagrams. Then, once this is done, we will study a very interesting non-relativistic example of a coupled theory in which non-relativistic fermions couple to phonons. This is the famous "electron-phonon" problem, and it provides a model for many field theories.

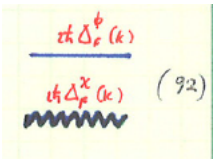
B.3.4(a) : RULES and DIAGRAMS for COUPLED SCALAR FIELDS

To simplify things we will consider 2 free scalar fields, which couple via a "cubic" coupling, linear in one variable and quadratic in the other; the action is then

$$S[\phi,\chi] \;=\; \int d^D x\, \frac{1}{2}\left\{\left[\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right] + \left[\partial_\mu\chi\partial^\mu\chi - M^2\chi^2\right] - \lambda_0\phi^2(x)\chi(x)\right\} \qquad (90)$$

so that the coupling vertex in this theory is $-i\lambda_0/2\hbar$. We can think of this theory as one in which the χ-particle is able to split into 2 ϕ-particles; alternatively, we can think of the χ-field as acting like an external force or source field acting on the ϕ-field. Each of these situations is shown in the diagrams - we imagine time flowing along the horizontal axis, in order to interpret these two diagrams.





From (90) we can easily write down a generating functional for this theory, and also the diagram rules. The generating functional is just

$$Z[J,I] \;=\; \int \mathcal{D}\phi \int \mathcal{D}\chi \; e^{\frac{i}{\hbar}\left\{S[\phi,\chi]+\int d^D x[J(x)\phi(x)+I(x)\chi(x)]\right\}} \tag{91}$$

where $J(x)$ couples to $\phi(x)$, and $I(x)$ to $\chi(x)$.

Now this is a very simple theory, as we can see from the Feynman rules, which are obviously just

1. Assign factors

$$\begin{aligned} G_0^{\phi}(k) &= i\hbar\Delta_F^{\phi}(k) \\ G_0^{\chi}(k) &= i\hbar\Delta_F^{\chi}(k) \end{aligned}$$

to the internal lines of a graph (92), where we have

$$\Delta_F^{\phi}(k) = \frac{1}{k^2 - m^2 + i\delta}, \qquad \Delta_F^{\chi}(k) = \frac{1}{k^2 - M^2 + i\delta} \tag{93}$$

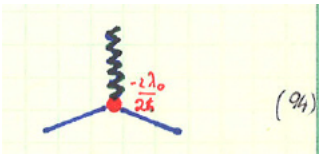
and all other conventions are as before.

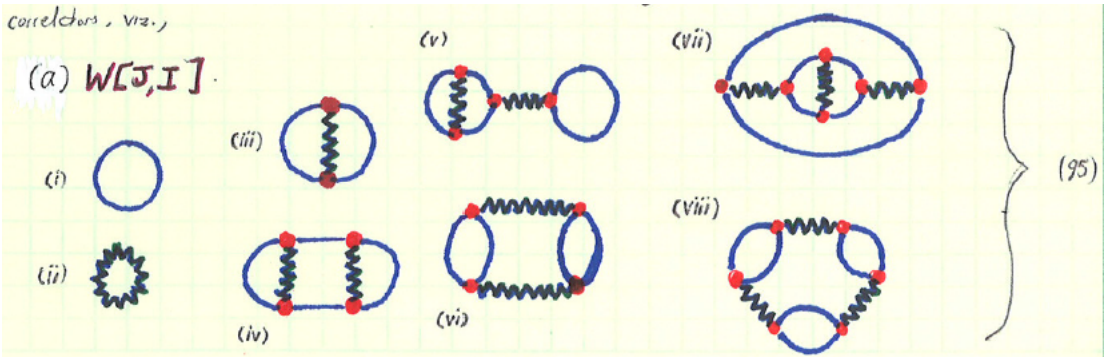
2. Assign a factor $-i\lambda/2\hbar$ to the vertex connecting the 2 fields, so we have a total contribution $(-i\lambda/2\hbar)\delta\big(\sum_j k_j\big)$ at each vertex, as shown in the diagram (94).

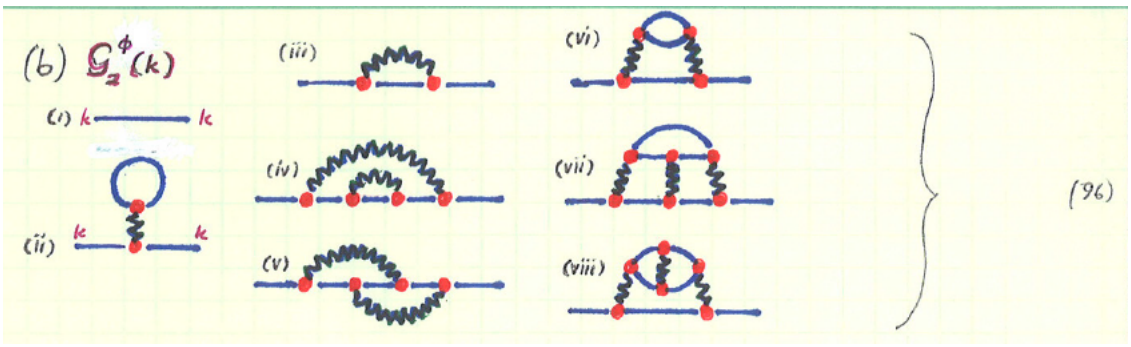
3. Integrate over all independent momenta.

4. Multiply each diagram by a symmetry factor.

Before looking at this theory from a functional point of view, let's just glance at the different kinds of graph we expect to find here. It is useful to make a classification of these according to the different connected correlators $\mathcal{G}_n^{(c)}(x_1,\ldots,x_n)$ and $\Gamma_n(x_1,\ldots,x_n)$ that we





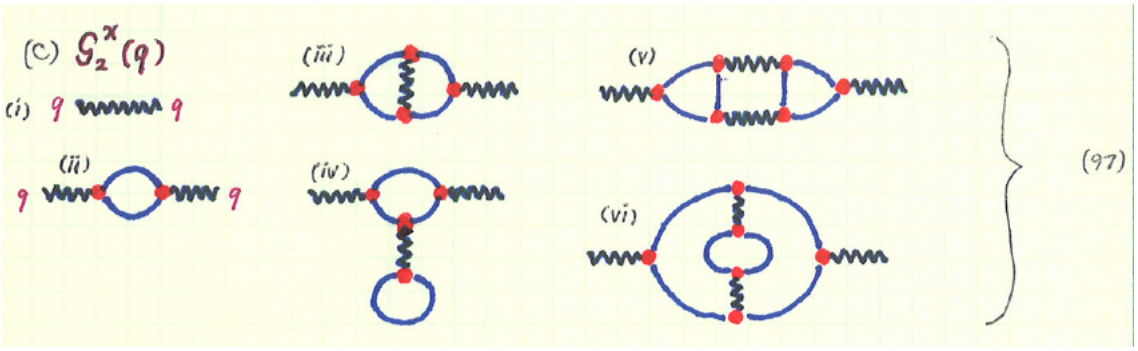


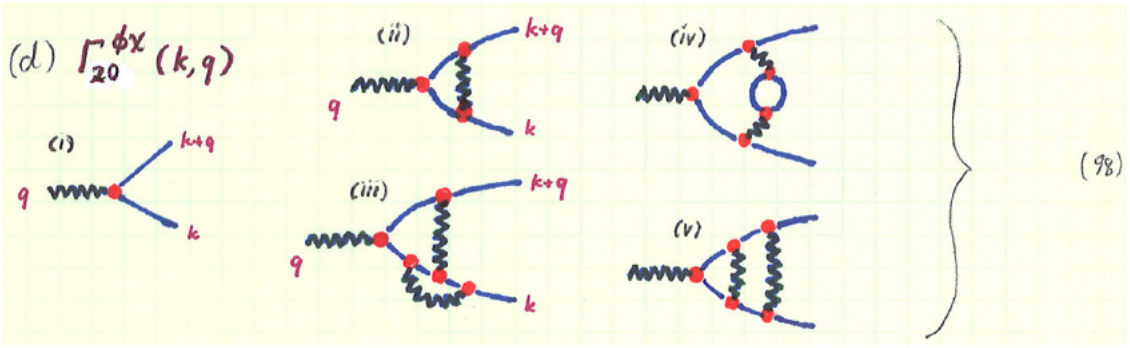
have seen before. In what follows I just show some graphs, with no attempt to evaluate them in any way. Let's begin with the generator $W[J, I]$ of connected correlators, as shown in the diagrams (95), (96) and (97),

These last 2 sets of graphs refer to the connected propagator for the ϕ-field and χ-field respectively. The first graph listed is the free graph, and then we go to successively higher graphs. Note that not all graphs at each order are shown - you can easily find the others. We will comment on structural features of these graphs later on.

The diagrams for what is usually called the "3-point vertex" show what are in effect corrections to the bare interaction vertex in the 1st graph (graph (98 (i))). These are usually called "vertex corrections". They also appear implicitly in the graphs shown above in (95)-(97) for W, $\mathcal{G}_2^{\phi}(k)$, and $\mathcal{G}_2^{\chi}(k)$.

From here we can go on to higher scattering correlators, and there is now a very large





choice. Before going on to enumerate graphs, let's just note how easy it is to formulate this mathematically. We can generate the usual expression for $W[J]$, a functional of a simple field source, to one which involves a pair of fields and a pair of sources - we simply expand (91) as

$$\begin{aligned} W[J,I] &= -i\hbar Z[J,I] \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)!}\left(\frac{-i}{\hbar}\right)^{n+m}\int d^Dx_1\cdots d^Dx_n\int d^Dy_1\cdots d^Dy_m \\ &\quad\times \mathcal{G}^{\phi\chi}_{nm}(x_1\ldots x_n,y_1\ldots y_m)\;J(x_1)\cdots J(x_n)\;I(y_1)\cdots I(y_m) \end{aligned} \tag{99}$$

thereby defining the correlators $\mathcal{G}^{\phi\chi}_{nm}(x_1\ldots x_n,y_1\ldots y_m)$, involving n external sources $J(x_k)$ and m external sources $I(y_k)$. In the same way we can define the generating functional for vertex parts as

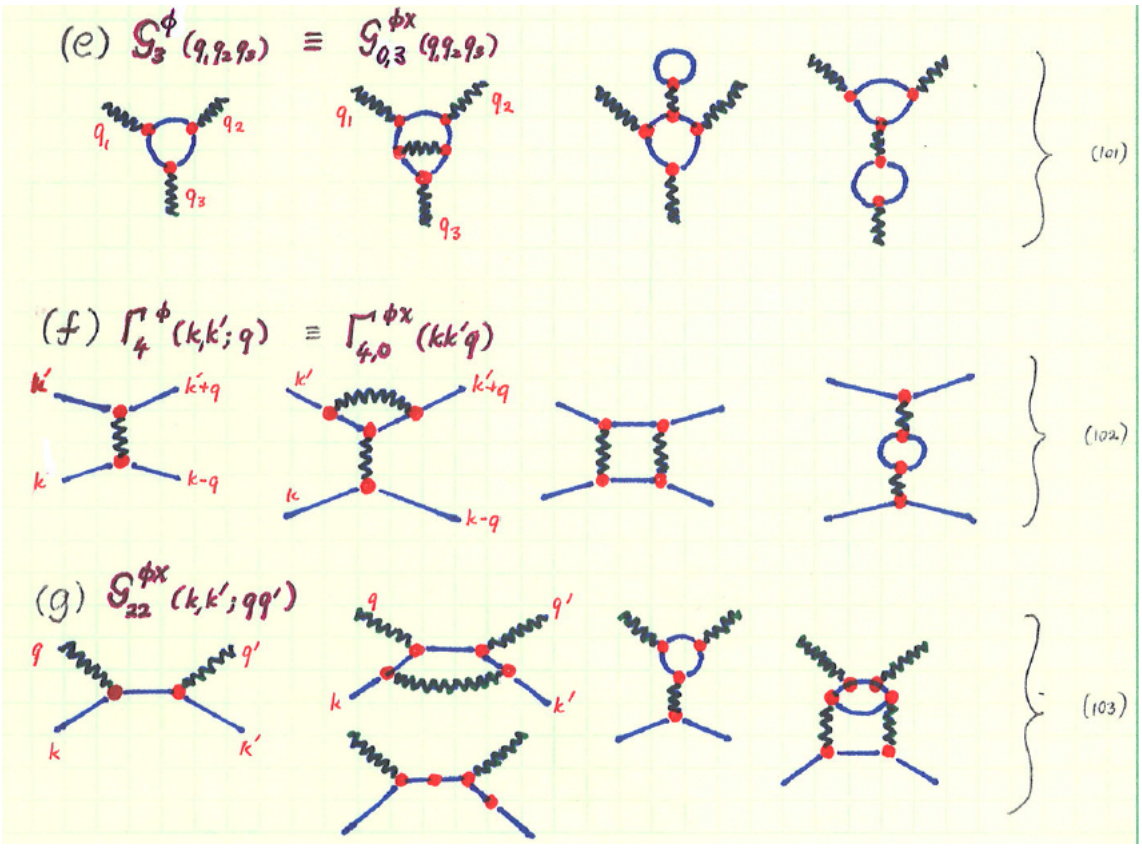
$$\begin{aligned} \Gamma[\phi,\chi] &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n+m)!}\int d^Dx_1\cdots d^Dx_n\int d^Dy_1\cdots d^Dy_m \\ &\quad\times \Gamma^{\phi\chi}_{n,m}(x_1\ldots x_n,y_1\ldots y_m)\;\phi(x_1)\cdots\phi(x_n)\;\chi(y_1)\cdots\chi(y_m) \end{aligned} \tag{100}$$

with n external ϕ-legs, and m external χ-legs.

Now let's look at some graphs for these - you should note that you could have written down these graphs without having (99) and (100) at your disposal (just as we did for (95)-(98)). And we can continue in this vein. Each of these diagrams has a definite physical meaning - this is one of the reasons for paying attention to them - and as you get used to them you will be able to interpret them.

Now let's look at all this from a functional point of view. We can't actually give a closed analytic solution to this problem, and so various approximation techniques can be applied - we will see some of these later. However, let's look here at a method that only involves things we've already seen. The idea is to first solve a simpler problem, in which one of the 2 fields is "frozen", and then see what happens when we unfreeze it.

B.3.4(b) : FREEZING $\chi(\mathbf{x})$ to a "BACKGROUND FIELD" $\chi_0(\mathbf{x})$


(e) $G_3^{\phi}(q_1 q_2 q_3) \equiv G_{0,3}^{\phi x}(q_1 q_2 q_3)$
q_1
q_2
q_3
(101)
(f) $\Gamma_4^{\phi}(k,k';q) \equiv \Gamma_{4,0}^{\phi x}(kk'q)$
k'
$k'+q$
k
$k-q$
(102)
(g) $G_{22}^{\phi x}(k,k';qq')$
q
q'
k
k'
(103)

Suppose we drop the external field $I(x)$ entirely, and "freeze" the field $\chi(x)$ (which is otherwise arbitrary) so that $\chi(x) \to \chi_0(x)$, some fixed function of x. The external current is now irrelevant here, because its purpose is to couple to $\chi(x)$, and change it.

If we do this, then our original problem, summarized in the double functional integral in (91), reduces now to an effective generating functional, i.e.,

$$\tilde{Z}_{\chi_0}[J] \;=\; \int \mathcal{D}\phi(x)\; e^{\frac{i}{\hbar}\left[\tilde{S}_{\chi_0}[\phi]+\int d^D x\; J(x)\phi(x)\right]} \tag{104}$$

with an effective action

$$\tilde{S}_{\chi_0}[\phi] \;=\; \int d^D x\; \frac{1}{2}\left[(\partial^\mu\phi\partial_\mu\phi) - \lambda_0\chi_0(x)\phi^2(x)\right] \tag{105}$$

Now it is immediately obvious that this problem is exactly solvable, since we are dealing with an effective action which is a quadratic form in $\phi(x)$. We will write down this solution in a minute, but it is worthwhile first looking at how to understand it physically. One way to do this is to notice that the last term in (105) is simply an addition to the effective mass of the field; we write

$$\begin{aligned} V_0(x) &= \frac{1}{2}\lambda_0\chi_0(x) \\ \tilde{m}^2(x) &= m_0^2 + V(x) \end{aligned} \tag{106}$$

i.e., we now have a spacetime dependent effective mass. But we can also think of $V(x)$ as a "scattering potential", which is dynamic (i.e., it depends on time as well as space), which can scatter the field $\phi(x)$ (and renormalize it as well). Either way, we see that $\chi_0(x)$ and $V(x)$ are behaving like a "background field" through which the field $\phi(x)$ propagates, while interacting with it. This is a simple example of a rather general method (called the "background field method") that we will meet again later.

Eqs. (104) and (105) are in the standard form $I = \int \mathcal{D}\phi\; e^{-\frac{1}{2}(\phi A_o\phi)-B_o\phi}$, with $A_o = \frac{i}{\hbar}\left[\partial^2 + m_0^2 + V_0(x)\right]$, $B_o = (i/\hbar)J$, giving $I = |A_o|^{-\frac{1}{2}} e^{-\frac{1}{2}(B_o A_o^{-1} B_o)}$, i.e., we have

$$\tilde{Z}_{V_0}[J] \;=\; \Big(\frac{\det(\partial^2+m_0^2)}{\det(\partial^2+m_0^2+V_0)}\Big)^{\frac{1}{2}}\; \exp\left[-\frac{i}{2\hbar}\int d^D x \int d^D x'\; J(x)\tilde{\Delta}\big(x,x' \mid V_0\big)J(x')\right] \tag{107}$$

where we have normalized $\tilde{Z}_{V_0}[J]$ so that when $V_0(x) \to 0$, we get back $\mathcal{Z}_0[J]$, i.e., we have divided by the free field determinant as we did for $\mathcal{Z}_0[J]$ (compare section B.1, eqs. (53) and (56)). The renormalized propagator $\tilde{\Delta}\big(x,x' \mid V_0\big)$ is the inverse of the renormalized differential operator, i.e.,

$$\big(\partial^2 + m_0^2 + V_0(x) - \delta\big)\; \tilde{\Delta}\big(x,x' \mid V_0\big) \;=\; -\delta(x-x') \tag{108}$$

i.e., $\tilde{\Delta}\big(x,x' \mid V_0\big)$ is the field propagator in the background field $V_0(x)$ (cf. section B.1, eq. (57)). This equation is easy to solve - it has the same structure as a simple QM scattering

problem, and we get

$$\tilde{\Delta}\big(x,x' \mid V_0\big) = \Delta_F(x,x') + \int d^D y \Delta_F(x,y) V_0(y) \tilde{\Delta}\big(y,x' \mid V_0\big) \tag{109}$$

or in k-space

$$\tilde{\Delta}\big(kk' \mid V_0\big) \;=\; \Delta_F(k)\,[1 - V_0(kk')\Delta_F(k)]^{-1} \tag{110}$$

If we employ the standard manouevre to change determinants to traces, by with

$$\det \hat{A} \;=\; \exp\Big\{\mathrm{Tr}\ln \hat{A}\Big\} \tag{111}$$

then we can write $\tilde{Z}_{V_0}[J]$ as

$$\tilde{Z}_{V_0}[J] \;=\; e^{\frac{i}{\hbar}\tilde{S}_{V_0}[J]} \tag{112}$$

where the "effective action" is

$$\tilde{S}_{V_0}[J] \;=\; \frac{1}{2}\left\{ i\hbar \mathrm{Tr}\ln\Big(\frac{\tilde{\Delta}\big(x,x' \mid V_0\big)}{\Delta_F(x,x')}\Big) - \int d^D x \int d^D x'\; J(x)\tilde{\Delta}\big(x,x' \mid V_0\big)J(x') \right\} \tag{113}$$

where the 1st term in $\tilde{S}_{V_0}[J]$ is often just incorporated into a prefactor, as in (107).

This result is easily interpreted both diagrammatically and physically. Let's first look at the diagrams.

(a) Prefactor: We can expand the "log" in the prefactor; provided the argument x is small, i.e., $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}x^n/n$, and so we can write the "Tr ln term as

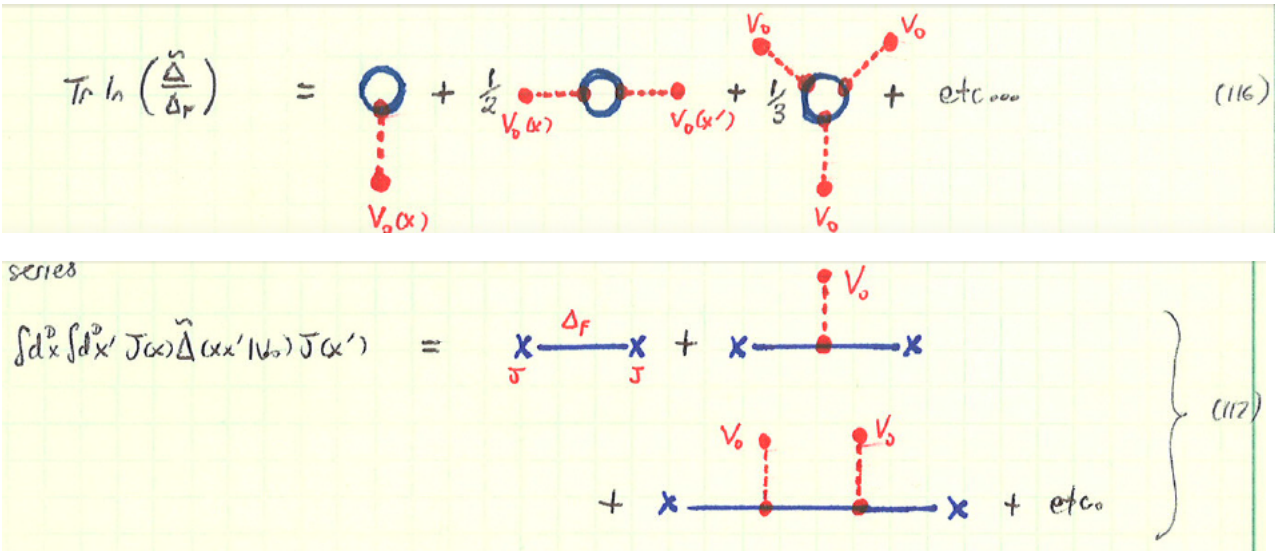
$$\begin{aligned}
\mathrm{Tr}\ln\Big(\frac{\tilde{\Delta}\big(x,x' \mid V_0\big)}{\Delta_F(x,x')}\Big) \;&=\; -\,\mathrm{Tr}\ln\Big[1 - V(x)\tilde{\Delta}_F(x,x')\Big] \\
&=\; -\int d^D x\, V(x)\Delta_F(0) \\
&\qquad +\; \frac{1}{2}\int d^D x \int d^D x'\; V(x)\Delta_F(x-x')V(x')\Delta_F(x'-x) + \ldots
\end{aligned} \tag{114}$$

leading to the result

$$\mathrm{Tr}\ln\Big(\frac{\tilde{\Delta}(x,x' \mid V_0)}{\Delta_F(x,x')}\Big) \;=\; \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\prod_{j=1}^{n}\int d^D x_j\; V(x_j)\Delta_F(x_j - x_{j+1})\delta(x_1 - x_n) \tag{115}$$

which we can show diagrammatically in a way similar to the diagrams for connected correlators, except that here the combinatorics are different; we have the series shown in the diagram (116)

(b) Exponent: The 2nd term in (113) has a form we've seen before. The key is that we have a propagator $\tilde{\Delta}(x,x' \mid V_0)$ which has external sources $J(x)$, $J(x')$ at its ends. Thus



we have a structure in which, according to (109) and (110), $V(x)$ acts repeated to give us a geometric series shown in the diagram (117), which shows graphically the repeated scattering of the free field off the background field, with interaction potential $V_0(x)$.

B.3.4(c) : REINTRODUCING ("UNFREEZING") the χ – FIELD:

In the diagrams we've just seen, and in the equations they refer to, we see that $V_0(x)$ plays the role of some "frozen" external field configuration - it does not itself get varied, and we do not care where it comes from.

However, "unfreezing" the χ-field means giving it back its dynamics (and also, if we wish, putting back the external driving field $I(x)$ acting on $\chi(x)$). Thus, the full generating functional $Z[J, I]$ for the coupled field system is (compare (90), (91), (104), and (105))

$$\mathcal{Z}[J, I] \;=\; \int \mathcal{D}\chi \; e^{\frac{i}{\hbar}\left(S_0[\chi] + \int d^D x \chi(x) I(x)\right)} \tilde{Z}_\chi[J] \tag{118}$$

where $\tilde{Z}_\chi[J]$ is just (112) with $V_0(x) = \frac{1}{2}\lambda_0\chi_0(x) \to \frac{1}{2}\lambda_0\chi(x)$, and $\chi(x)$ is now free to vary. The interpretation of (118) is interesting - it is as though we were evaluating the generating functional for the χ-field, except that now every configuration $\chi(x)$ is weighed by a factor $\tilde{Z}_\chi[J]$. The great advantage of this result from the QFT point of view is that it is exact and non-perturbative in form; there is no expansion in coupling constants. We can, if we wish, generate a diagrammatic expression for $Z[J, I]$ from (118), using the usual techniques - this will give us back the diagrams we saw in previous pages for $Z[J, I]$, expanded in λ_o.

However there are many other ways of generating diagrammatic expressions, and just to give a taste of one of them, and also to introduce a useful trick, I briefly discuss here another approach, which relies not on taking functional derivatives with respect to external currents $J(x)$ and/or $I(x)$, but instead with respect to the internal fields themselves.

To do this, let's just briefly look again at the formula derived in section B.1, eqs. (81)-(83). There is another way of getting the result in (82) which illustrates the use of "shift operators". As discussed in Appendix on functionals, the identity

$$e^{a(d/dx)} f(x) \quad = \quad f(x+a) \tag{119}$$

generalizes to

$$e^{\int dx\ A(x)(\delta/\delta\psi(x))}\ F[\psi(x)] \quad = \quad F[\psi(x)+A(x)] \tag{120}$$

when we deal with functionals. So let's apply this idea now to the partition function or generating functional. We have, for a simple ϕ-field, the result

$$\begin{aligned} \mathcal{Z}[J] \quad &= \mathcal{N} \int \mathcal{D}\phi\ e^{\frac{i}{\hbar}\left(S_0[\phi]+\int d^D x\ (J(x)\phi(x)-V[\phi])\right)} \\ &= \mathcal{N} \int \mathcal{D}\phi e^{\frac{i}{\hbar}S_0[\phi]}\ e^{\int d^D x\ \phi(x)\delta/\delta\psi(x)}\ e^{\frac{i}{\hbar}\int d^D x\ \left(J(x)\psi(x)-V[\psi]\right)}\ |_{\psi=0} \end{aligned} \tag{121}$$

i.e., we use (120) and then set $\psi = 0$ at the end to get $F[A]$; this is a neat trick for changing variables (NB: $\mathcal{N}N$ is just a normalizing factor).

However, we notice now that the first 2 terms in (121) can be written

$$\begin{aligned} I \quad &= \quad \int \mathcal{D}\phi\ e^{\frac{i}{\hbar}S_0[\phi]}\ e^{\int d^D x\ \phi(x)\delta/\delta\psi(x)} \\ &= \quad \int \mathcal{D}\phi\ e^{\frac{-i}{2\hbar}\int d^D x\ \phi(x)(\partial^2+m^2)\phi(x)}\ e^{\int d^D x\ \phi(x)\delta/\delta\psi(x)} \\ &= \quad \exp\frac{i}{\hbar}\int d^D x \int d^D x'\ \frac{1}{2}\delta/\delta\psi(x)\ \Delta_F(x-x')\ \delta/\delta\psi(x') \quad \equiv \quad Z_0[-i\hbar\delta/\delta\psi(x)] \end{aligned} \tag{122}$$

so that we can rewrite $\mathcal{Z}[J]$ as

$$\mathcal{Z}[J] \quad \propto \quad \mathcal{Z}[-i\hbar\delta/\delta\psi(x)]\ e^{\frac{i}{\hbar}\int d^D x\ \left(J(x)\psi(x)-V[\psi]\right)}\ |_{\psi=0} \tag{123}$$

which is just eq. (82) of section B.1, derived in a different way, using the "shift operator" in (120).

Now let's use this trick on our results for the coupled field problem. In (118) we are only interested in the external field $I(x)$, coupling to $\chi(x)$, if we want to look at correlation functions involving $\chi(x)$. Let's now think of $\chi(x)$ as some kind of "quantum environment" for the central field $\phi(x)$; we will be interested in what $\phi(x)$ does, and so all we want to do is to integrate out $\chi(x)$, and assume it is otherwise passive.

This means we are interested in

$$\mathcal{Z}[J,0] \quad = \quad \int \mathcal{D}\chi\ e^{\frac{i}{\hbar}S_0[\phi]}\ \tilde{Z}_\chi[J] \tag{124}$$

which is going to tell us how the ϕ-field behaves, once we have functionally averaged over the χ-field.

Now we do the same as above, writing

$$\begin{aligned} Z[J,0] &= \int \mathcal{D}\chi(x)\; e^{\frac{i}{\hbar}S_0[\phi]}\; e^{d^Dx\;\chi(x)\delta/\delta\psi(x)}\tilde{Z}_\psi[J]\mid_{\psi=0} \\ &= e^{\frac{i\hbar}{2}\int d^Dx\;\int d^Dx'\;\delta/\delta\psi(x)\;\Delta_F(x-x')\;\delta/\delta\psi(x')}\;\tilde{Z}_\psi[J]\mid_{\psi=0} \end{aligned} \tag{125}$$

where $\Delta_F^\chi(x,x')$ is the Feynman propagator for the free χ-field (with mass M, not m_0). Substituting in $\tilde{Z}_\chi[J]$ from (112) and (113), we get the exact expression:

$$\begin{aligned} Z[J,0] &= e^{\frac{i\hbar}{2}\int d^Dx\;\int d^Dx'\;\delta/\delta\psi(x)\;\Delta_F(x-x')\;\delta/\delta\psi(x')} \\ &\times\; e^{-\frac{1}{2}\mathrm{Tr}\ln\left(\frac{\tilde{\Delta}(x,x'\mid\psi)}{\Delta_F(x,x')}\right)}\;\exp\frac{-i}{2\hbar}\int d^Dx\;\int d^Dx'\;J(x)\;\tilde{\Delta}(x,x'\mid\psi)\;J(x')\mid_{\psi=0} \end{aligned} \tag{126}$$

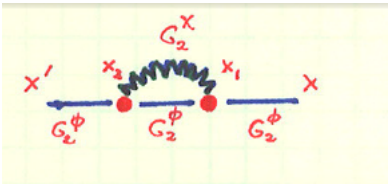
Although this expression is exact, it is also rather forbidding. However, its physical meaning is interesting. As written in (125), and spelled out explicitly in (126), we see that we are operating on the diagrams in $\tilde{Z}_\psi[J]$, shown in eqs. (116) and (117), with an operator $\exp\left\{\frac{i}{2\hbar}\int \delta/\delta\psi\;\Delta_F\;\delta/\delta\psi\right\}$ which has the effect of linking or joining up all the external χ-lines which are attached to the ϕ-field lines in (116) and (117). Thus all the "dangling" χ-fields that are attached to the external fields in (116) and (117) get joined up or "sewn together" to produce a result of $\mathcal{Z}[J,0]$.

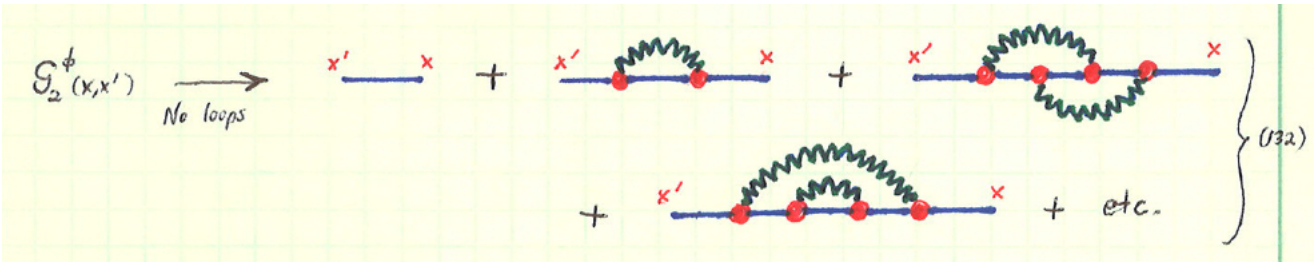
To see how this works, let's do something with (126). One of the most useful things to calculate is the 2-point function $\mathcal{G}_2^{(\phi)}(x,x')$, which tells us how the ϕ-field propagates - we already saw some diagrams for this in eq. (96). This is just given, in the usual way, by the 2nd functional derivative of $\mathcal{Z}[J,0]$ with respect to $J(x)$, taken at $J=0$. We can easily do this, but the result is rather messy. However, we can also do an expansion in "loops" of $\mathcal{Z}[J,0]$ in (126) (we will look in more detail later at loop expansions, which are equivalent to semiclassical expansions - we already mentioned them in the previous pages of this section). All the loops in (126) are coming from the "prefactor" in the expression for $\tilde{Z}_\psi[J]$, i.e., from the $\exp\left[-\frac{1}{2}\mathrm{Tr}\ln(\tilde{\Delta}/\Delta_F)\right]$ term - compare eq. (116). So if simply suppress this term entirely (i.e., make it equal to unity) then we will get all the graphs for $\mathcal{Z}[J,0]$ without loops. The result turns out to be quite elegant. Starting from

$$\mathcal{Z}[J,0] \qquad \longrightarrow \qquad e^{\frac{i\hbar}{2}\int d^Dx\int d^Dx'\;\frac{\delta}{\delta\psi(x)}\;\Delta_F^\chi(x-x')\;\frac{\delta}{\delta\psi(x')}}\;\; e^{\frac{-i}{2\hbar}\int d^Dx\int d^Dx'\;J(x)\;\tilde{\Delta}(x,x'|\psi)\;J(x')}\mid_{\psi=0} \tag{4}$$
$$\text{(no loops)}$$

we immediately find that

$$\mathcal{G}_2^{(\phi\phi)}(x,x') \qquad \longrightarrow \qquad e^{\frac{i\hbar}{2}\int d^Dx\int d^Dx'\;\frac{\delta}{\delta\psi(x)}\;\Delta_F^\chi(x-x')\;\frac{\delta}{\delta\psi(x')}}\;\;\tilde{G}_2(x,x'\mid\psi)\mid_{\psi=0} \tag{5}$$
$$\text{(no loops)}$$





where

$$\tilde{G}_2(x,x' \mid \psi) \;=\; i\hbar\tilde{\Delta}_2(x,x' \mid \psi) \tag{129}$$

with $\tilde{\Delta}_2(x,x' \mid \psi)$ the Feynman propagator in the presence of the field $\psi(x)$, i.e., the propagator satisfying

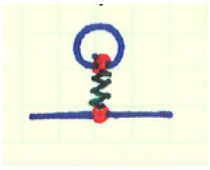
$$\left(\partial^2 + m_0^2 + \frac{1}{2}\lambda_0\psi(x)\right)\tilde{\Delta}_2(x,x' \mid \psi) \;=\; -\delta(x-x') \tag{130}$$

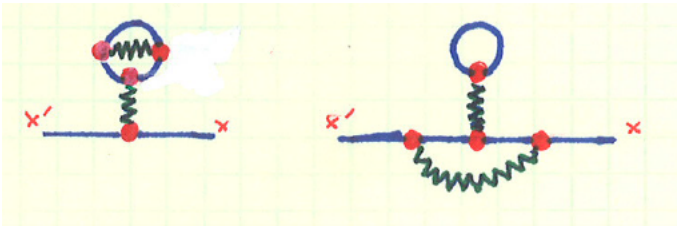
(compare eq. (108)). Now if we expand the exponential in (128), we see that its effect is to operate an even number of times on $\tilde{G}_2(x,x' \mid \psi)$, and each time it does this, it produces another pair of lines for the ϕ-field, joined by the interaction line $\Delta_F^\chi(x,x')$, with coupling constant $\frac{1}{2}\lambda_0$ at each vertex. Thus, e.g.,

$$\begin{aligned}
&\frac{i\hbar}{2}\frac{\delta}{\delta\psi(x)}\,\Delta_F^\chi(x-x')\,\frac{\delta}{\delta\psi(x')}\,\tilde{G}_2(x,x' \mid \psi)\,|_{\psi=0} \\
&\quad= \frac{-\lambda_0^2}{4}(ik)^2\int d^Dx_1\int d^Dx_2\Delta_F(x-x_1)\Delta_F(x_1-x_2)\Delta_F(x_2-x')\Delta_F^\chi(x_1-x_2) \\
&\quad= \left(\frac{i\lambda_0}{2}\right)^2\int d^Dx_1\int d^Dx_2\; G_2^\phi(x-x_1)G_2^\phi(x_1-x_2)G_2^\phi(x_2-x')G_2^\phi(x_1-x_2)
\end{aligned} \tag{131}$$

which is shown at the top. It is then easily seen that the expansion of (128) gives the series of graphs shown in the figure (132) - we have no loops:

We can also see, if we look again at (116), how we can add back the loops into this expansion. This, if we include only the 1st term in (116), we are adding back "tadpole" diagrams into $\mathcal{G}_2^\phi(x,x')$, the 1st one of which is shown at the bottom. Repeated application of the functional derivatives, according to (128), will then generate all possible combinations of tadpoles with non-loop interactions, all mediated by the χ-field propagator $\tilde{G}_2(x,x') = i\hbar\tilde{\Delta}_F^\chi(x,x')$ given in eq. (93). Two such diagrams, both 4th-order in the coupling constant, are shown at the top. By adding in more loop terms, we see that we can generate all possible graphs for $\mathcal{G}_2^\phi(x,x')$.





We see that we have now found a way of calculating the correlators for any kind of coupled field problems, in some kind of diagrammatic expansion. In later sections we will show different sorts of expansions, based on these exact results, to get useful answers.

B.3.5: The COUPLED ELECTRON-PHONON SYSTEM

As we have already seen, real phonons in real solids are complicated. The main reason for this is that one has both transverse and longitudinal phonons, even in a homogeneous isotropic solid, and far more complicated modes in a crystal. This means that the study of electron-phonon interactions in a real solid is very technically demanding, and of no special interest in a general course of QFT. For those interested in this, there are quite a few books and reviews that deal with it.

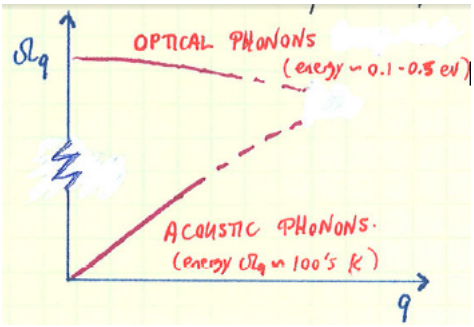
3.2.5 (a) TOY MODEL for ELECTRON-PHONON INTERACTIONS

There is a "toy model" that is often employed to study electron-phonon interactions, which is of some interest as a problem in the QFT of coupled fields. In this model we make the following assumptions:

(a) The Hamiltonian of the system has a simple electron-phonon coupling with no structure - in fact the Hamiltonian is taken to be

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \hbar\Omega_{\mathbf{q}} (b^\dagger_{\mathbf{q}} b_{\mathbf{q}} + \frac{1}{2}) + \lambda_0 \int d^3\mathbf{r} \psi^\dagger_\sigma(\mathbf{r}) \psi_\sigma(\mathbf{r}) \phi(\mathbf{r}) \qquad (133)$$

where $\phi(\mathbf{r})$ is the phonon amplitude previously defined in (74), and $\psi^\dagger_\sigma(\mathbf{r})$, $\psi_\sigma(\mathbf{r})$ are the Fourier transforms of $c^\dagger_{\mathbf{k}\sigma}$, $c_{\mathbf{k}\sigma}$, and are electron creation and annihilation operators. We have already known the Feynman rules for the electrons and phonons here; they were given in (81)-(83), and in (88) and (89). Thus all we are adding to this is the coupling vertex,



which amounts to the rules:

$$\text{Phonon propagator:} \quad D_0(\mathbf{q},\omega) \;=\; i\hbar \frac{\Omega_{\mathbf{q}}^2}{\omega^2 - (\Omega_{\mathbf{q}} - i\delta)^2} \tag{134}$$

$$\text{Electron propagator:} \quad G_0^{\sigma\sigma'}(\mathbf{k},\epsilon) \;=\; i\hbar\delta_{\sigma\sigma'} \left[\frac{\theta(\mathbf{k}-\mathbf{k}_F)}{\hbar\epsilon - E_{\mathbf{k}}^0 + i\delta} + \frac{\theta(\mathbf{k}_F - \mathbf{k})}{\hbar\epsilon - E_{\mathbf{k}}^0 - i\delta} \right] \tag{135}$$

where we define

$$E_{\mathbf{k}}^0 \;=\; \epsilon_{\mathbf{k}}^0 - \mu \tag{136}$$

and electron-phonon vertex:

$$-\frac{i\lambda_)}{\hbar}\, \delta\Big(\sum_j \mathbf{q}_j + \mathbf{k}_j\Big)\, \delta\Big(\sum_j \epsilon_j + \omega_j\Big) \tag{137}$$

(b) The dispersion relation $\Omega_{\mathbf{q}}$ for the electrons will be assumed to be quite simple. First, in this toy model one ignores the difference between longitudinal and transverse phonons, just assuming simple longitudinal phonons. We then assume there are only 2 phonon branches, as shown at the top, as follows:

(i) Acoustic Phonons: This is just ordinary sound, with a dispersion relation

$$\Omega_{\mathbf{q}} \;=\; c_0 \mathbf{q} + \mathcal{O}(\mathbf{q}^2) \tag{138}$$

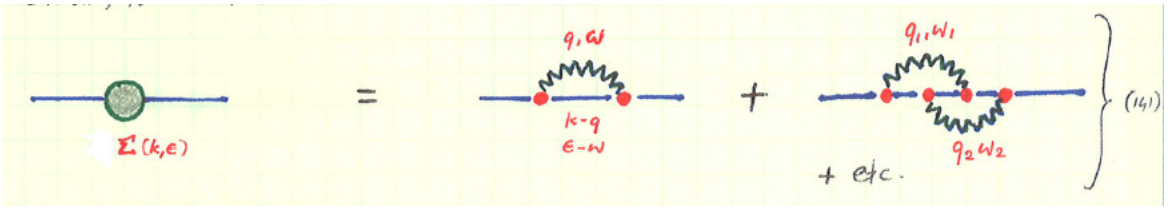
where c_0 is the sound velocity; the higher corrections come from both the lattice structure and from phonon-phonon couplings;

(ii) Optical Phonons: These are high frequency oscillations that do not exist in a liquid, but arise in solid because one can have anti-phase oscillations between neighboring ions. Then the dispersion relation is

$$\Omega_{\mathbf{q}} \;=\; \Omega_0 + \mathcal{O}(\mathbf{q}^2) \tag{139}$$

with Ω_0 the optical phonon energy gap; typically the term $\sim \mathcal{O}(\mathbf{q}^2)$ has a negative coefficient.

With this simple model one can understand some rather important aspects of the electron-phonon interaction.



Note, incidentally, that in texts and in the literature you will often find different expressions from (134) for the propagator. The reason for this is that while we have been using the "density-density" correlator (cf. eqs. (74)-(78)), many authors instead use the "phase-phase" correlator $\mathcal{D}_0^{\phi\phi}(\mathbf{r}\mathbf{r}', tt')$, defined by

$$\mathcal{D}_0^{\phi\phi}(\mathbf{r}\mathbf{r}', tt') \;=\; \langle 0 \mid \hat{T}\left\{\phi(\mathbf{r}, t)\phi(\mathbf{r}', t')\right\} \mid 0\rangle \tag{140}$$

which gives a different result for $D_0(\mathbf{q}, \omega)$. For those who are interested, the appendix on Feynman rules and Feynman diagrams gives details of these different definitions.

Now let's look briefly at the sort of calculations one can do with models like this. I will consider 2 cases, each of which has an enormous literature devoted to it, we will barely scratch the surface.

B.3.5 (b) ELECTRON-PHONON INTERACTIONS in a METAL

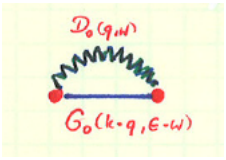
In this system we have a finite-density electron systems interacting with phonons - only the interaction with acoustic phonons is important. A proper treatment takes account of the electron-electron interactions as well, but we ignore these for the moment. Let's now look at some basic results for this system.

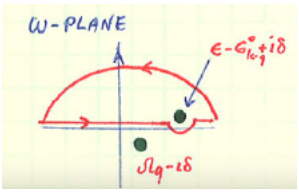
(i) Electron Self-Energy: As we saw back in section B.1, we can separate off a "self-energy part" from the total electron Green function $\mathcal{G}_2(\mathbf{k}, \epsilon)$; see eqs. (47)-(49) of section B.1. We have already seen some graphs for this in eq. (96) of this section - for the electron-phonon problem we have the graphs shown, to lowest order:

Now let's evaluate the lowest-order graph shown here (NB: the "tadpole graph", shown in (96)(ii), is zero for the electron-phonon system). This is given by

$$\begin{aligned} i\Sigma(\mathbf{k}, \epsilon) \;&=\; \left(\frac{-i\lambda_0}{\hbar^2}\right)^2 \sum_{\mathbf{q}} \int \frac{d\omega}{2\pi}\, G_0(\mathbf{k}-\mathbf{q}, \epsilon-\omega) D_0(\mathbf{q}, \omega) \\ &=\; i\lambda_0^2 \sum_{\mathbf{q}} \int \frac{d\omega}{2\pi i}\, \Omega_{\mathbf{q}} \left[\frac{1}{\omega - \Omega_{\mathbf{q}} + i\delta} - \frac{1}{\omega + \Omega_{\mathbf{q}} - i\delta}\right] \\ &\qquad \times \left[\frac{1 - f_{\mathbf{k}-\mathbf{q}}}{\epsilon - \omega - \epsilon^o_{\mathbf{k}-\mathbf{q}} + i\delta} + \frac{f_{\mathbf{k}-\mathbf{q}}}{\epsilon - \omega - \epsilon^o_{\mathbf{k}-\mathbf{q}} - i\delta}\right] \end{aligned} \tag{142}$$

where we have used the notation $f_{\mathbf{k}} = f(\epsilon^o_{\mathbf{k}})$; note that $f_{\mathbf{k}} \to \theta(\mathbf{k}_F - |\mathbf{k}|)$ for the Fermi function at $T = 0$. Now we just need to do a contour integral to get the frequency integration done -





there are 2 contributions to this integral:

(a) The contribution coming from the product with the 2 poles shown, in the ω-plane, is

$$\int \frac{d\omega}{2\pi i} \frac{1}{\omega - \Omega_{\mathbf{q}} + i\delta} \frac{1 - f_{\mathbf{k}-\mathbf{q}}}{\epsilon - \omega - \epsilon^o_{\mathbf{k}-\mathbf{q}} + i\delta} = \frac{1 - f_{\mathbf{k}-\mathbf{q}}}{\epsilon - \epsilon^o_{\mathbf{k}-\mathbf{q}} - \Omega_{\mathbf{q}} + i\delta} \tag{143}$$

where we have used Cauchy's theorem, i.e., that

$$f(z) = \oint \frac{d\omega}{2\pi i} \frac{f(\omega)}{\omega - z} \tag{144}$$

and closed the integral as shown in the diagram, circling below the pole at $\omega = \epsilon - \epsilon^o_{\mathbf{k}-\mathbf{q}} + i\delta$.

(b) The other contribution is

$$\int \frac{d\omega}{2\pi i} \frac{1}{\omega + \Omega_{\mathbf{q}} - i\delta} \frac{f_{\mathbf{k}-\mathbf{q}}}{\epsilon - \omega - \epsilon^o_{\mathbf{k}-\mathbf{q}} - i\delta} = \frac{f_{\mathbf{k}-\mathbf{q}}}{\epsilon - \epsilon^o_{\mathbf{k}-\mathbf{q}} - \Omega_{\mathbf{q}} - i\delta} \tag{145}$$

circling the pole at $\epsilon = -\Omega_{\mathbf{q}} + i\delta$.

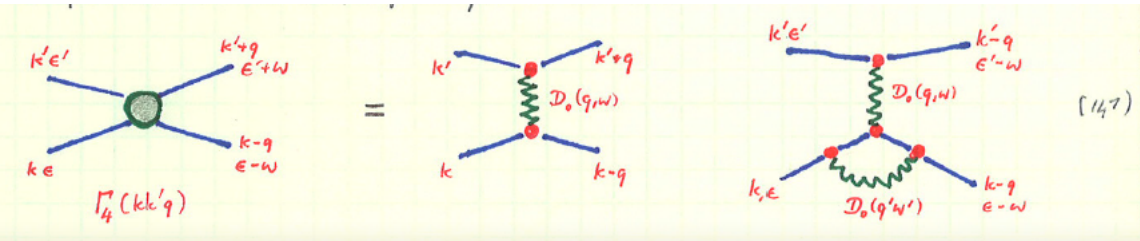
The other 2 contributions to $\Sigma(\mathbf{k}, \sigma)$ give zero - both poles are then on the same side of the real axis (both above, or both below) and so the contour integration gives zero.

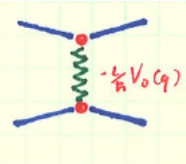
The final result is then

$$\Sigma(\mathbf{k}, \sigma) = \lambda_0^2 \sum_{\mathbf{q}} \Omega_{\mathbf{q}} \left[\frac{1 - f_{\mathbf{k}-\mathbf{q}}}{\epsilon - \epsilon^o_{\mathbf{k}-\mathbf{q}} - \Omega_{\mathbf{q}} + i\delta} + \frac{f_{\mathbf{k}-\mathbf{q}}}{\epsilon - \epsilon^o_{\mathbf{k}-\mathbf{q}} + \Omega_{\mathbf{q}} - i\delta} \right] \tag{146}$$

in which the first term corresponds to the intermediate state with an electron in a particle state above the Fermi surface, and the 2nd to where it is in a hole state below the Fermi surface.

Actually, because the phonon energies are so small, the corrections coming from $\sum(\mathbf{k}, \sigma)$ to the electron energy $\epsilon^o_{\mathbf{k}}$ are only appreciable near the Fermi surface.





(ii) Phonon-mediated Electron-Electron interaction: Let's now look at the 4-point vertex for electron-electron scattering via phonons. We have seen the diagrams in eq. (102); for the electron-phonon problem we have the diagram (147)

Now let's write this as $\Gamma_4(\mathbf{k}\mathbf{k}'\,\mathbf{q}) = -\frac{i}{\hbar}V_{eff}(\mathbf{k}\mathbf{k}'\,\mathbf{q})$, where the interaction $V_{eff}(\mathbf{k}\mathbf{k}'\,\mathbf{q})$ is a renormalized version of the bare interaction λ_0. Then the lowest contribution is

$$\begin{aligned} -\frac{i}{\hbar}\,V_{eff}(\mathbf{k}\mathbf{k}'\,\mathbf{q}) \;&\rightarrow\; -\frac{i}{\hbar}V_0(\mathbf{q}) \\ &= \left(\frac{i}{\hbar}\right)^2 D_0(\mathbf{q},\omega) \;=\; -\frac{i}{\hbar}\lambda_0^2\,\frac{\Omega_{\mathbf{q}}^2}{\omega^2-(\Omega_{\mathbf{q}}-i\delta)^2} \end{aligned} \tag{148}$$

and we see that

$$V_0(\mathbf{q},\omega) \;=\; \lambda_0^2\frac{\Omega_{\mathbf{q}}^2}{\omega^2-(\Omega_{\mathbf{q}}-i\delta)^2} \;\rightarrow\; -\lambda_0^2 \qquad (\text{if } \omega \ll \Omega_{\mathbf{q}}) \tag{149}$$

a result which is crucial to the theory of conventional electron-phonon superconductors, where the crucial energies are low. In real superconductors, electron-electron interactions partially screen this attractive interaction, but not fully.

There are corrections to this result - the 2nd diagram in (147) changes the effective vertex λ_0 in $V_0(\mathbf{q},\omega)$. The change is

$$-\frac{i}{\hbar}\delta\Lambda(\mathbf{k},\mathbf{q}) \;=\; \left(-\frac{i\lambda_0}{\hbar}\right)^3\sum_{\mathbf{q}'}\int\frac{d\omega'}{2\pi}\;\left\{G_0(\mathbf{k}-\mathbf{q}',\epsilon-\omega')D_0(\mathbf{q}',\omega')G_0(\mathbf{k}+\mathbf{q}-\mathbf{q}',\epsilon+\omega-\omega')\right\} \tag{150}$$

which is actually fairly messy to evaluate. In the standard theory of the electron-phonon interaction this contribution is rather crucial - it was first evaluated by Migdal, who showed it to give a small correction to the bare vertex. This is not because λ_0 is small, but because the phonon energies $\ll$ the Fermi energy. Thus the vast majority of intermediate electrons do not feel the correction, and the two diagrams we have just calculated turn out to be the key contributions.

B.3.5 (c): ELECTRON-PHONON INTERACTIONS in an INSULATOR

Let's now look very briefly at what happens to an electron in an insulator or a semiconductor. Because there is an energy gap, the coupling to high-energy optical phonons turns out to be more important in many cases.

Now from the diagram shown above, we see that the optical phonons are much more like Einstein phonons - almost no dispersion in **q**-space, and so quasi-localized in real space. It is then useful to compare the real space and momentum space representations of both the interaction and the diagrams. Let's rewrite the Hamiltonian in (133) in a "site" basis, assuming that all electronic states are quasi-localized around sites j. Now let's try to do a slightly more general analysis then that embodied in (133). We label the phonon states by indices λ (with no commitment yet as to what λ might be), and write the total Hamiltonian in the form

$$H \;=\; -\sum_{ij}\sum_{\sigma} t_{ij}\big(\{b_\lambda\}\big)c^\dagger_{i\sigma}c_{j\sigma} + \sum_\lambda \Omega_\lambda b^\dagger_\lambda b_\lambda + \sum_{j\sigma}\epsilon_j\big(\{b_\lambda\}\big)c^\dagger_{j\sigma}c_{j\sigma} \tag{147}$$

so that we have a "hopping term" t_{ij} between electronic sites, and where the local energy on site j is ϵ_j; both are now functionals of the phonon variables b_λ, $b^\dagger_\lambda$. This formalism can be used even for a random system. A common approximation makes the following 2 assumptions:

(i) an ordered crystalline lattice;

(ii) no phonons (ie., we drop the dependence on the $\{b_\lambda\}$;

and in this case we have:

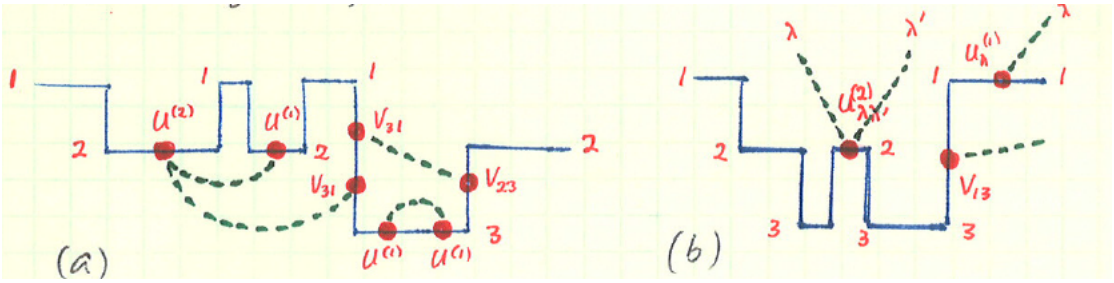
$$t_{ij}\big(\{b_\lambda\}\big) \;\rightarrow\; t^o_{ij}; \qquad\qquad \epsilon^o_{\mathbf{k}} = \sum_j t^o_{ij} e^{i\mathbf{k}\cdot(\mathbf{r}_j - \mathbf{r}_i)} \tag{148}$$

where $\mathbf{r}_i, \mathbf{r}_j$ label sites, and t^o_{ij} depends only on $\mathbf{r}_j - \mathbf{r}_i$; the energy $\epsilon^o_{\mathbf{k}}$ is just the band energy for this lattice.

More generally, we can drop these assumptions - ie., bring back the phonons and assume an arbitrary lattice. In this case t^o_{ij} becomes some arbitrary function of the site indices i and j (and we can no longer define a band energy), and we can expand the local energies in powers of the phonon variables:

$$\begin{aligned}\epsilon_j\big(\{b_\lambda\}\big) \;&=\; \epsilon^0_j \;+\; \sum_\lambda u^{(1)}_j(\lambda)[b_\lambda + b^\dagger_\lambda] \\ &\qquad +\;\; \frac{1}{2}\sum_{\lambda\lambda'} u^{(2)}_j(\lambda,\lambda')[b_\lambda + b^\dagger_\lambda][b_{\lambda'} + b^\dagger_{\lambda'}] \;+\; \cdots \end{aligned} \tag{149}$$

$$t_{ij}\big(\{b_\lambda\}\big) \;=\; t^0_{ij}\;\exp\left\{-\sum_\lambda \frac{V_{ij}(\lambda)}{\Omega_\lambda}[b_\lambda + b^\dagger_\lambda]\right\} + \cdots \tag{150}$$



where in (150) we recognize that the tunnelling amplitude $t_{ij}\big(\{b_\lambda\}\big)$ will be influenced by phonons mainly by the alteration of the tunnelling exponent (the phonons change the distance between the 2 sites, thereby altering the electronic tunneling amplitude between orbitals on the 2 sites). The terms in (149) correspond to multi-phonon couplings to the electron when it is on site j; the terms in (150) act only when the phonon is hopping between sites. We can show them diagrammatically, as shown in the figure below.

The diagram (b) at right shows the interactions involved in (149) and (150). The vertex $u_j^{(2)}(\lambda,\lambda')$ emits or absorbs 2 phonons when the electron is sitting on some site - the sites are labelled in the figures by indices $1,2,3$, with time along the horizontal axis. The vertex $u_j^{(1)}(\lambda)$ has a single on site interaction; but the vertex $V_{ij}(\lambda)$ emits or absorbs a phonon λ when the electron is hopping from site j to site i. Typically we stop the exponential expansion at the first term, i.e., we write

$$t_{ij}\big(\{b_\lambda\}\big) \quad \sim \quad t_{ij}^0\left[1-\sum_\lambda \frac{V_{ij}}{\Omega_\lambda}[b_\lambda+b_\lambda^\dagger]\right] \tag{151}$$

Now let's suppose we have a regular lattice. Then we can Fourier transform all this back to $\mathbf{k}$-space, to get

$$H \;=\; \sum_{\mathbf{k}\sigma}\epsilon^o_{\mathbf{k}}c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+\sum_{\mathbf{q}}\Omega_{\mathbf{q}}b^\dagger_{\mathbf{q}}b_{\mathbf{q}}+\sum_{\mathbf{kq}}\sum_\sigma g(\mathbf{k},\mathbf{q})c^\dagger_{\mathbf{k}+\mathbf{q},\sigma}c_{\mathbf{k}\sigma}[b_{\mathbf{q}}+b^\dagger_{\mathbf{q}}] \;+\; \text{etc.} \tag{152}$$

where we have only included the $u^{(1)}$ and $V_{ij}^{(2)}$ terms, and the interaction term is the Fourier transform of the site interaction

$$H \;=\; \sum_j u_j^{(1)}c^\dagger_{i\sigma}c_{j\sigma}(b_j+b_j^\dagger) \;+\; \sum_{ij}t^o_{ij}\frac{V_{ij}}{\Omega_0}\Big[c_i^\dagger c_j(b_i+b_j^\dagger)+\text{H.c.}\Big] \tag{153}$$

so that

$$g(\mathbf{k},\mathbf{q}) = g_1(\mathbf{q})+g_2(\mathbf{k},\mathbf{q}) \tag{154}$$

in which $g_1(\mathbf{q})$ is the Fourier transform of the first interaction term in (153), and $g_2(\mathbf{k},\mathbf{q})$ transforms the 2nd term; and we have put $\Omega_\lambda\to\Omega_0$ for all $\mathbf{q}$.

Without going into any details, we can already see how different this case is from a metal. In the first place, Ω_0 is rather big, so that we can't just stick to low-order diagrams.

A simplifying feature comes because often we are dealing with only one, or a very few, electrons. Then the self-energy simplifies considerably. Consider, e.g., the 1st-order graph we already studied. Now it becomes

$$\Sigma(\mathbf{k},\epsilon) \;=\; \sum_{\mathbf{q}} |g(\mathbf{k},\mathbf{q})|^2 \frac{1}{\epsilon - \epsilon^0_{\mathbf{k}+\mathbf{q}} - \Omega_0 + i\delta} \tag{155}$$

because now $f_{\mathbf{k}-\mathbf{q}} = 0$ (there is no Fermi surface, and $k_F = 0$). The energies Ω_0 and t^o_{ij} may well be comparable here, and so if $g(\mathbf{k},\mathbf{q})$ is not small, there is no reason for higher graphs to be neglected.

For many years the only sensible way of getting answers to this "polaron model" was to use non-perturbative path integral methods - summing diagrams was essentially useless. However in recent years, very powerful numerical methods have become available, and these have allowed a quantitative solution to the polaron system.

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This concludes our initial snapshot of perturbative methods. Some key things that have been left out are:

(i) Finite-temperature perturbation theory, and graphical rules for QFT at finite T. These are particularly important in condensed matter systems.

(ii) Perturbation theory and Feynman rules for more complex theories, e.g., spin systems, lattice systems, disordered systems, and strongly-correlated systems in condensed matter theory; and systems of higher spin in relativistic QFT.

(iii) Gauge theories - these include QED, QCD or Yang-Mills theory and the standard model and quantum gravity, as well as a host of condensed matter examples.

(iv) More general relations that hold between field-theoretic objects, both in perturbation theory and beyong; these include Schwinger-Dyson equations, Ward identities, and so on.

(v) Other non-perturbative approximation techniques.

In future chapters we will be filling in these gaps.