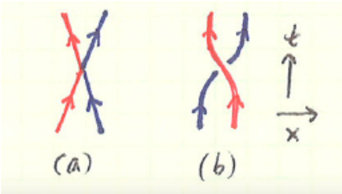

B2. Fermionic Path Integrals

Whether we look at many-particle systems in terms of operators or in terms of path integrals, there is a fundamental fact in Nature that we have to face up to, i.e., the existence of quantum statistics and the difference this imposes on the path existing in many-fermion (as opposed to many-boson) systems.

To see why there must be a crucial difference in the paths, it suffices to consider the two processes shown in the figure.

In (a) we see 2 identical particles pass through the some spacetime point. As we know this is perfectly OK for bosons, but not for fermions (the exclusion principle). And in the 2nd diagram (b), 2 fermions exchange positions, a manoeuvre which does nothing to the boson pair wave-function, but which multiplies the fermion pair wave-function by a factor -1.

The way to deal with fermionic path integrals was found by Berezin in 1964, using a Grassmann variable formulation. In what follows I explain this in a very simple way. A better job can be done by using coherent state path integrals, but this would take us a little far afield. Then, by way of example, the path integral propagator is given for non-interacting Dirac fermions; and we discuss the interacting non-relativistic electron gas.

B.2.1 GENERATING FUNCTIONAL for FERMIONS

Let us first recall the basic operator algebra for fermionic operators c_k, $c_k^\dagger$; we have the results:

$$\begin{aligned}
\{c_k, c_{k'}\} &= c_k c_{k'} + c_{k'} c_k = 0 \\
\{c_k, c^\dagger_{k'}\} &= c_k c^\dagger_{k'} + c^\dagger_{k'} c_k = \delta_{kk'} \\
c_k^2 &= (c_k^\dagger)^2 = 0
\end{aligned} \tag{1}$$

so that $c_k c_{k'} = -c_{k'} c_k$ and $c_k c^\dagger_{k'} = \delta_{kk'} - c^\dagger_{k'} c_k$; recall also that the number operator for fermions is

$$n_k = c_k^\dagger c_k \tag{2}$$

Now let's consider an algebra of "Grassmann variables" $\eta_j = (\eta_{j_1}, \eta_{j_2}, \ldots, \eta_{j_n})$, i.e., Grassmann variable with n components. The simplest one to consider is the 1-dimensional variable or Grassmann number η, and we specify that

$$\{\eta, \eta\} = 0 \qquad (\text{so} \quad \eta^2 = 0) \tag{3}$$

as a consequence of which any function $f(\eta)$ can be written

$$f(\eta) = f_0 + \eta f_1 \tag{4}$$

since all terms $\approx O(\eta^2)$ are zero. Now define "left" and "right" derivatives of these, such that

$$\frac{\overrightarrow{d}}{d\eta}\eta = \frac{d}{d\eta}\eta = 1, \qquad \eta\frac{\overleftarrow{d}}{d\eta} = -1 \tag{5}$$

$$\frac{d}{d\eta}f(\eta) = f_1, \qquad f(\eta)\frac{\overleftarrow{d}}{d\eta} = -f_1 \tag{6}$$

While sticking with the 1-d Grassmann numbers, let's ask how this works if we have a whole set $\{\eta_j\}$ of these numbers. Then it is clear that we will have

$$\{\eta_i, \eta_j\} = 0 \tag{7}$$

so that $\eta_j^2 = 0$; in the same way, a general function will be of the form $f(\eta_j) = a_j + b_j\eta_j$, and so on. The derivatives will now obey

$$\begin{aligned} \frac{\overrightarrow{d}}{d\eta_i}(\eta_j\eta_k) &= \delta_{ik}\eta_j - \delta_{ij}\eta_k \\ (\eta_j\eta_k)\frac{\overleftarrow{d}}{d\eta_i} &= \delta_{ij}\eta_k - \delta_{ik}\eta_j \end{aligned} \tag{8}$$

and the analogue of the $[\rho, x]$ commutator for bosons is

$$\left\{\frac{d}{d\eta_i}, \frac{d}{d\eta_j}\right\} \;=\; \delta_{ij} \tag{1}$$

which for 1-d Grassmann numbers becomes:

$$\left\{\frac{d}{d\eta}, \eta\right\} \;=\; 1 \tag{9}$$

and finally we have

$$\left\{\frac{d}{d\eta_i}, \frac{d}{d\eta_j}\right\} = 0 \tag{10}$$

so that $\left(\frac{d}{d\eta}\right)^2 = 0$.

This last result, that $\frac{d}{d\eta}\frac{d}{d\eta} = 0$, indicates that to define integration over Grassmann variable is not so obvious, since it apparently means that the operation of differentiation does not have an inverse integration operation. However we now argue that any sensible definition of integration must make an integral independent of any change of variable, i.e., that

$$\int_{-\infty}^{\infty} d\eta f(\eta) = \int_{-\infty}^{\infty} d\eta f(\eta + a) \quad \Longrightarrow \quad \begin{aligned} \int d\eta \;&=\; 0 \\ \int d\eta\eta \;&=\; 1 \end{aligned} \tag{11}$$

where the latter is really just a normalization convention. Comparing eqtns. (5) and (11), we see that integration and differentiation do the same things to a function $f(\eta)$, i.e., they are the same operation; it follows that

$$\int d\eta \int d\eta \;=\; \int d\eta \left(\frac{d}{d\eta}\right) \;=\; \frac{d^2}{d\eta^2} = 0 \tag{12}$$

Finally, let us look at the analogue of conventional Jacobian, produced by changing variables in an integration. Suppose we change variable as follows:

$$\eta \to \zeta \qquad \text{and let}: \quad \eta \;=\; a\zeta + b \tag{13}$$

Now, our definition of integration will be

$$\int d\eta f(\eta) \;\equiv\; \frac{d}{d\eta} f(\eta) \tag{14}$$

so that if we start from the integral

$$\begin{aligned} I &= \int d\eta\; f(\eta) \\ &= \int d\eta\; f(a\zeta + b) \end{aligned} \tag{15}$$

we also have, by definition, that

$$\begin{aligned} I \;=\; \frac{d}{d\eta} f(a\zeta + b) \;&=\; \left(\frac{d\zeta}{d\eta}\right) \frac{d}{d\zeta} f(a\zeta + b) \\ &=\; \frac{1}{a}\frac{d}{d\zeta} f(a\zeta + b) \;=\; \frac{1}{a}\int d\zeta f(a\zeta + b) \end{aligned} \tag{16}$$

The key point to notice here is that the answer is not what we would have naively derived by a straightforward change of variable inside the integral; in other words, the manoeuvre

$\int d\eta f(\eta) \;\to\; \int d\zeta \frac{d\eta}{d\zeta} f(a\zeta + b) \;=\; a\int d\zeta f(a\zeta + b) \;= a\int d\zeta f(a\zeta + b)$ is **WRONG**!

We see here how we must always define integration according to eq. (15); thus, the Jacobian is the inverse of what we would expect. The result in eq. (16) is easily generalized to a multi-dimensional integral over Grasssmann variable, to give

$$I \;=\; intd\theta_1 \ldots \theta_n f(\theta_1, \ldots, \theta_n) \;=\; \int d\zeta_1 \ldots d\zeta_n \; J_{\theta\zeta}^{-1} \; f(\zeta_1, \ldots, \zeta_n) \tag{17}$$

where the Jacobian $J_{\theta\zeta}$ is

$$J_{\theta\zeta} = \left| \frac{\partial \theta_i}{\partial \zeta_j} \right| \tag{18}$$

a result which is easily proved by induction; and in the same way, we can write that if

$$\theta_i \;=\; M_{ij}\zeta_j \qquad \Longrightarrow \qquad \prod_{i=1}^{n} \theta_i \;=\; |M_{ij}| \prod_{j=1}^{n} \zeta_j \tag{19}$$

where $|M_{ij}| = \det M_{\theta\zeta}$.

Typically we will be dealing with exponentials of quadratic forms in Grassmann variables; i.e., Gaussian functions of Grassmann variables, of form

$$I(A) \;=\; \prod_{j=1}^{n} \int d\bar{\eta}_j d\eta_j \exp \left[\sum_{kl} \bar{\eta}_k A_{kl} \eta_l \right] \;\to\; \prod_{j=1}^{n} \int d\bar{\eta}_j d\eta_j \; \prod_{k=1}^{n} \prod_{l=1}^{n} (1 + \bar{\eta}_k A_{kl} \eta_l) \tag{20}$$

where the last form comes from expanding the exponential as in eq. (4). We can see how things will work for an arbitrary matrix A_{kl} by looking at the 2-dimensional case. Let $\eta = (\eta_1 \; \eta_2)$, so that

$$\bar{\eta}\eta \;=\; (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2) \tag{21}$$

(where we think of this as multiplying a column vector by its complex row transpose); also

$$(\bar{\eta}\eta)^2 \;=\; 2\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 \tag{22}$$

and higher powers of $\bar{\eta}\eta$ are zero; consequently we have

$$\int d\bar{\eta} \int d\eta \; e^{-\bar{\eta}\eta} \;=\; \int d\bar{\eta}_1 \int d\bar{\eta}_2 \int d\eta_1 \int d\eta_2 \; (\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2) \;=\; 1 \tag{23}$$

and so for the 2-d Gaussian integral we find

$$\int d\bar{\eta}_1 \int d\eta_1 \int d\bar{\eta}_2 \int d\eta_2 \; \exp \left[\sum_{k,l=1}^{2} \bar{\eta}_k A_{kl} \eta_l \right] \;=\; (A_{11}A_{22} - A_{21}A_{12}) \;=\; \det|\mathbf{A}| \tag{24}$$

and in the n-dimensional case

$$\int d\bar{\eta} \int d\eta \; e^{-(\eta, A\eta)} \;=\; \sum_{j=1}^{n} \int d\bar{\eta}_j \int d\eta_j e^{\sum_{kl} \bar{\eta}_k A_{kl} \eta_l} \;=\; \det|\mathbf{A}| \tag{25}$$

Actually in physical applications we will be interested in Grassmann-variable integrals in which external sources acting on fields $\bar{\psi}$ and ψ will be introduced. If we denote the sources themselves by $\bar{\eta}$ and η, then the kind of integral we will be evaluating has the form, in n dimensions, of an integral

$$I_n(\bar{\eta},\eta) \;=\; \prod_{j=1}^{n} \int d\bar{\psi}_j \int d\psi_j \exp\left\{ \sum_i \sum_j \bar{\psi}_i A_{ij} \psi_j \cdot \sum_j (\bar{\psi}_j \eta_j + \bar{\eta}_j \psi_j) \right\} \tag{26}$$

and with simple change of variable (shift)

$$\psi_j = \chi_j + \sum_k A^{-1}_{jk} \eta_k, \qquad \bar{\psi}_j = \bar{\chi}_j + \sum_k \bar{\eta}_k A^{-1}_{kj} \tag{27}$$

we then integrate over the χ_j and $\bar{\chi}_j$, having diagonalized the quadratic form in (26). The result is

$$I_n(\bar{\eta},\eta) \;=\; e^{-\sum_{jk} \bar{\eta}_j A^{-1}_{jk} \eta_k} \ \det|\mathbf{A}| \tag{28}$$

So far so good - we now have all the results we need to deal with fermionic path integrals. It simply remains to generalize these results to the infinite dimensional Hilbert spaces appropriate to a QFT, retaining the rules of integration over Grassmann variables that we have found. We therefore introduce fermionic field variable $\psi(x)$, $\bar{\psi}(x)$, which are postulated to satisfy relations analogous to (1)-(13), ie.,

$$\begin{aligned} &\{\eta(x),\eta(x')\} = \{\bar{\eta}(x),\bar{\eta}(x')\} = \{\bar{\eta}(x),\eta(x')\} = 0 \\ &\{\eta(x),\psi(x')\} = \{\bar{\eta}(x),\bar{\psi}(x')\} = \{\bar{\eta}(x),\psi(x')\} = \{\eta(x),\bar{\psi}(x')\} = 0 \end{aligned} \tag{29}$$

and

$$\left\{ \frac{\delta}{\delta\bar{\eta}(x)}, \bar{\eta}(x') \right\} = \left\{ \frac{\delta}{\delta\eta(x)}, \eta(x') \right\} \;=\; \delta(x-x') \tag{30}$$

$$\left\{ \frac{\delta}{\delta\eta(x)}, \bar{\eta}(x') \right\} = \left\{ \frac{\delta}{\delta\bar{\eta}(x)}, \eta(x') \right\} = \left\{ \frac{\delta}{\delta\eta(x)}, \frac{\delta}{\delta\eta(x')} \right\} \;=\; 0 \tag{31}$$

and finally

$$\begin{aligned} &\int d\psi(x) = 0 \\ &\int d\psi(x)\ \psi(x) = 1 \end{aligned} \tag{32}$$

Now, with all these preliminaries accomplished, we can turn to a fermionic field theory. Let's first look at the non-interacting case, and consider a free field Lagrangian of form (with sources):

$$L_0(\bar{\psi},\psi;\bar{\eta},\eta) \;=\; \bar{\psi}(x) G_0^{-1}(x-x')\psi(x') \;+\; (\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)) \tag{33}$$

where $G_0(x-x')$ is the inverse of the "kinetic energy" or free field operator that appears in the Lagrangian (we write it in this way because we will find that $G_0(x-x')$ is the free field propagator, as expected). To be quite general we have written G_0^{-1} as if it were non-local in x-space.

Now consider the form the partition function or generating functional will take. We wish it to be properly normalized, so we will write it in the form

$$\mathcal{Z}_0[\bar{\eta},\eta] = \left\langle 0 \middle| \hat{T} \exp\left\{ \frac{i}{\hbar} \int d^4x \left[\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right] \right\} \middle| 0 \right\rangle \tag{34}$$

$$= \frac{\int \mathcal{D}\bar{\psi} \int \mathcal{D}\psi \ \exp \frac{i}{\hbar} \left\{ \int d^4x \int d^4x' \ \bar{\psi}(x) G_0^{-1}(x,x')\psi(x') + \int d^4x \left[\bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x) \right] \right\}}{\int \mathcal{D}\bar{\psi} \int \mathcal{D}\psi \ \exp \frac{i}{\hbar} \int d^4x \int d^4x' \ \bar{\psi}(x) G_0^{-1}(x,x')\psi(x')} \tag{35}$$

Now we know from eqs. (25) and (28) how to evaluate the numerator and denominator of eq. (35). The determinants cancel, and so we just get

$$\mathcal{Z}_0[\bar{\eta},\eta] = \exp \frac{-i}{\hbar} \int d^4x \int d^4x' \ \bar{\eta}(x) G_0(x-x')\eta(x') \tag{36}$$

a form which you will immediately recognize, as being like that for a simple 1-particle Green function subject to a noise "source". The only difference here, and from the scalar field case, is the presence of 2 sources $\bar{\eta}(x)$ and $\eta(x)$ (with their Grassmann commutation relations).

In the same way as for ordinary QM, or for ϕ^4 theory, we can also deal with interactions. Suppose we now have a Lagrangian

$$L(\bar{\psi},\psi;\bar{\eta},\eta) = L_0(\bar{\psi},\psi;\bar{\eta},\eta) - V(\bar{\psi},\psi) \tag{37}$$

with L_0 given by eq. (33) and $V(\bar{\psi},\psi)$ some self-interaction term (the simplest form being a "ϕ^4" theory, with $V(\bar{\psi},\psi) = V_0(\bar{\psi}\psi)^2$). Then it is easy to show that

$$\mathcal{Z}[\bar{\eta},\eta] = \exp\left\{ -\frac{i}{\hbar} \int d^4x \ V\left(i\hbar \frac{\delta}{\delta\bar{\eta}}, -i\hbar \frac{\delta}{\delta\eta} \right) \right\} Z_0[\bar{\eta},\eta] \tag{38}$$

where $V\left(i\hbar \frac{\delta}{\delta\bar{\eta}}, -i\hbar \frac{\delta}{\delta\eta} \right)$ is just $V(\bar{\psi},\psi)$, we have replaced $\bar{\psi}$ by $i\hbar\frac{\delta}{\delta\bar{\eta}}$, and replaced ψ by $-i\hbar\frac{\delta}{\delta\eta}$. This is derived as before, by writing everything as a polynomial expansion.

Consider now the form taken by the correlation functions. Let us define

$$G_{2n}(x_1,\dots,x_n,x_1',\dots x_n') = \langle 0 \mid \hat{T} \left\{ \psi(x_1)\cdots\psi(x_n) \quad \bar{\psi}(x_1')\cdots\bar{\psi}(x_n') \right\} \mid 0 \rangle \tag{39}$$

with the $\bar{\psi}'$s to the right of ψ's. Note that these correlation functions must satisfy the antisymmetry requirements

$$\begin{aligned} G_{2n}(x_1,x_2,\dots) &= -G_{2n}(x_2,x_1,\dots) \\ G_{2n}(\dots,x_1',x_2',\dots) &= -G_{2n}(\dots,x_2',x_1',\dots) \end{aligned} \tag{40}$$

But we can also write these correlation functions in terms of functional derivatives of $\mathcal{Z}[\bar{\eta}, \eta]$, just as we did for ϕ^4 theory. We assume that all functional derivative operators act to the right in what follows, so that from eq. (31) we have

$$\begin{aligned}
\left\{ \frac{\delta}{\delta\bar{\eta}(x)}, \bar{\eta}(x') \right\} = \left\{ \frac{\delta}{\delta\eta(x)}, \eta(x') \right\} &= \delta(x - x') \\
\left\{ \frac{\delta}{\delta\bar{\eta}}, \eta \right\} = \left\{ \frac{\delta}{\delta\eta}, \bar{\eta} \right\} &= 0 \\
\left\{ \frac{\delta}{\delta\bar{\eta}}, \frac{\delta}{\delta\bar{\eta}} \right\} = \left\{ \frac{\delta}{\delta\eta}, \frac{\delta}{\delta\eta} \right\} &= 0
\end{aligned} \tag{41}$$

Let us now apply this to functional derivatives of $\mathcal{Z}[\bar{\eta}, \eta]$. We see that we have

$$\langle \psi(x) \rangle \;=\; \langle 0 \mid \hat{T} \left\{ \psi(x) e^{\frac{i}{\hbar} \int d^4x \left[\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right]} \right\} \mid 0 \rangle \;=\; -i\hbar \frac{\delta Z}{\delta\bar{\eta}(x)} \tag{42}$$

$$\langle \bar{\psi}(x) \rangle \;=\; \langle 0 \mid \hat{T} \left\{ \bar{\psi}(x) e^{\frac{i}{\hbar} \int d^4x \left[\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right]} \right\} \mid 0 \rangle \;=\; i\hbar \frac{\delta Z}{\delta\eta(x)} \tag{43}$$

and we note the key difference of sign. However for any correlator like eq. (39), with an equal number of ψ and $\bar{\psi}$ operators, the sign is always positive - you should check this - and we get

$$G_{2n}(x_1, \ldots, x_n, x_1^{'}, \ldots x_n^{'}) \;=\; \hbar^{2n} \frac{\delta^{2n} Z}{\delta\bar{\eta}(x_1) \ldots \bar{\eta}(x_n) \delta\eta(x_1^{'}) \ldots \eta(x_n^{'})} \Big|_{\bar{\eta},\eta=0} \tag{44}$$

B.2.2 EXAMPLE: DIRAC FERMIONS

Our first example comes from relativistic QM or QFT, although one of the more interesting realizations of the idea comes from the non-relativistic graphene systems - we have a Lagrangian

$$L_0(\bar{\psi}, \psi; \bar{\eta}, \eta) \;=\; \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) + [\bar{\eta}\psi + \bar{\psi}\eta] \tag{45}$$

where the γ^μ are the Dirac γ-matrices for a relativistic spin-$\frac{1}{2}$ electron, satisfying

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu} \\
\gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3
\end{aligned} \tag{46}$$

so that γ^5 anticommutes with the others:

$$\{\gamma^5, \gamma^\mu\} \;=\; 0 \tag{47}$$

Note that there are various different conventions in common use for the γ-matrices, involving different representation for them. [1] The one we will use writes the 4×4 matrices as:

$$\begin{aligned}\gamma^0 &= \hat{\tau}_z \otimes \hat{I} \\ \gamma^j &= i\hat{\tau}_y \otimes \hat{\sigma}_j \\ \gamma^5 &= \hat{\tau}_x \otimes \hat{I}\end{aligned} \tag{48}$$

where the $\{\tau_j\}$, $\{\hat{\sigma}_j\}$ are Pauli matrices, and $\hat{I}$ is the unit matrix. Since this is not a course in particle physics, I will not go into any more detail here on the Dirac equation. In relativistic QED the Dirac operators act on 4-d spinors. It is quite illustrating to refer to the condensed matter application of these spinors, where the $\hat{\tau}_j$ operate in "particle-hole" space (so that, eg., $\hat{\tau}_x$ changes a particle to a hole, or vice-versa), and the $\hat{\sigma}_j$ operate in spin space (so that $\hat{\sigma}_x$ flips a spin from $|\uparrow\rangle$ to $|\downarrow\rangle$ or vice-versa).

Now from eq. (36) we can write the partition function for this system as

$$\mathcal{Z}_0[\bar{\eta}, \eta] \;=\; \exp\left\{ \frac{-i}{\hbar} \int d^4x \int d^4x' \; \bar{\eta}(x) \left[(i\gamma^\mu \partial_\mu + m) \Delta_F(x - x') \right] \eta(x') \right\} \tag{49}$$

where $\Delta_F(x)$ is just the Feynman propagator, defined as before by

$$(\partial^2 + m^2)\Delta_F(x) \;=\; -\delta^4(x) \tag{50}$$

and with the usual representation

$$\Delta_F(x) \;=\; \sum_k \frac{e^{ikx}}{k^2 - m^2 + i\delta} \tag{51}$$

To prove that

$$G_0(x) \;=\; (i\gamma^\mu \partial_\mu + m)\Delta_F(x) \tag{52}$$

as assumed in eq. (49), it suffices to evaluate

$$\begin{aligned}G_0^{-1}(x) G_0(x) &= (i\gamma^\mu \partial_\mu - m)(i\gamma^\mu \partial_\mu + m)\Delta_F(x) \\ &= -(\partial^2 + m^2)\Delta_F(x) \;=\; \delta^4(x)\end{aligned} \tag{53}$$

which verifies eq. (52). Note that $G_0(x)$ is not quite the propagator for the Dirac field - we can find this by evaluating the functional differential of $Z_0[\bar{\eta}, \eta]$ in eq. (49) to find

$$G_2^0(x - x') \;=\; \hbar^2 \frac{\delta^2 Z_0[\bar{\eta}, \eta]}{\delta\bar{\eta}(x)\delta\eta(x')} \Big|_{\bar{\eta},\eta=0} \;=\; i\hbar G_0(x - x') \tag{54}$$

Note that in k-space the Dirac fermion free propagator takes a much simpler form - one simply has

$$G_2(k) \;=\; i\hbar G_0(k) \;=\; \frac{i\hbar}{\gamma^\mu k_\mu - m + i\delta} \tag{55}$$

[1] Another common convention for the γ-matrices has $\gamma^0 = \hat{\tau}_x \otimes \hat{I}$, $\gamma^j = -i\hat{\tau}_y \otimes \hat{\sigma}_j$, and $\gamma^5 = \hat{\tau}_z \otimes \hat{I}$.

Later on we will see how these Dirac fermions can be used to describe a whole variety of relativistic and non-relativistic systems. We will also add interactions - this leads many important applications.

B.2.3 EXAMPLE: ELECTRON FLUID

The 2nd example is one that is central to condensed matter physics, just as Dirac fermions are central to QED. Thus we now deal with non-relativistic fermions, for which a separation between space - this variable is made - we no longer have Lorentz invariance.

Consider a set of non-relativistic fermions - these could be electrons, or some other set of fermions like ^{3}He atoms. Now in ordinary QM we would simply start from a Hamiltonian of form

$$H \;=\; \sum_{j=1}^{N} \frac{-\hbar^2}{2m} \nabla_j^2 \;+\; \frac{1}{2} \sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j) \qquad (56)$$

but this is not what we want for a field theory. The next step is then to invent 2nd-quantized field operators satisfying:

$$\begin{aligned} \{\hat{c}_\sigma(\mathbf{r}), \hat{c}^\dagger_{\sigma'}(\mathbf{r}')\} &= \delta_{\sigma\sigma'}\delta(\mathbf{r} - \mathbf{r}') \\ \{\hat{c}^\dagger_\sigma(\mathbf{r}), \hat{c}^\dagger_{\sigma'}(\mathbf{r}')\} &= \{\hat{c}_\sigma(\mathbf{r}), \hat{c}_{\sigma'}(\mathbf{r}')\} \;=\; 0 \end{aligned} \qquad (56')$$

where $\sigma = \pm$ labels the spin projection; and we want these operators to act on a Fermi vacuum state according to

$$\hat{c}_\sigma(\mathbf{r})|0\rangle \;=\; 0 \qquad\qquad \langle 0|\hat{c}^\dagger_\sigma(\mathbf{r}) \;=\; 0 \qquad (57)$$

Since the electron system here is spatially homogeneous, according to eqtns. (56) and (56'), it is better to go over to D-dimensional momentum space, so that we define

$$\begin{aligned} \hat{c}_{\mathbf{k},\sigma} &= \int d^D r\, \hat{c}_\sigma(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \\ \hat{c}^\dagger_{\mathbf{k},\sigma} &= \int d^D r\, \hat{c}^\dagger_\sigma(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \end{aligned} \qquad (58)$$

and so then

$$\begin{aligned} \{\hat{c}_{\mathbf{k},\sigma}, \hat{c}^\dagger_{\mathbf{k}',\sigma'}\} &= \delta_{\sigma\sigma'}\delta_{\mathbf{k}\mathbf{k}'} \\ \{\hat{c}^\dagger_{\mathbf{k},\sigma}, \hat{c}^\dagger_{\mathbf{k}',\sigma'}\} &= \{\hat{c}_{\mathbf{k},\sigma}, \hat{c}_{\mathbf{k}',\sigma'}\} \;=\; 0 \end{aligned} \qquad (59)$$

and we can now write a Hamiltonian in terms of these 2nd-quantized operators as

$$\begin{aligned} \mathcal{H} &= \frac{-\hbar^2}{2m} \int d^D r\, \hat{c}^\dagger_\sigma(\mathbf{r}) \nabla^2 \hat{c}_\sigma(\mathbf{r}) \;+\; \frac{1}{2} \int d^D r \int d^D r'\, \hat{c}^\dagger_\sigma(\mathbf{r})\hat{c}_\sigma(\mathbf{r}) V(\mathbf{r} - \mathbf{r}') \hat{c}^\dagger_{\sigma'}(\mathbf{r}')\hat{c}_{\sigma'}(\mathbf{r}') \\ &= \sum_{\mathbf{k},\sigma} \epsilon^o_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k},\sigma}\hat{c}_{\mathbf{k},\sigma} + \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}'} \sum_{\mathbf{q}} \sum_{\sigma,\sigma'} \hat{c}^\dagger_{\mathbf{k}+\mathbf{q},\sigma}\hat{c}_{\mathbf{k},\sigma} V(\mathbf{q}) \hat{c}^\dagger_{\mathbf{k}',\sigma'}\hat{c}_{\mathbf{k}'-\mathbf{q}',\sigma'} \end{aligned} \qquad (60)$$

where $V(\mathbf{q}) = \int d^D r\, V(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}}$, and $\epsilon^o_{\mathbf{k}} = \hbar^2 k^2/2m$.

However this is still not what we want - although eq. (60) defines a specific non-relativistic field theory, we want to write everything in terms of path integrals. To do this one can go through the usual "Trotter product" manoeuvre that we used for 1-particle mechanics; the result is now written as

$$\mathcal{Z}[\bar{\eta}, \eta] \;=\; \int \mathcal{D}\,\psi^*_{\mathbf{k}\sigma} \int \mathcal{D}\psi_{\mathbf{k}\sigma} \exp\left\{ \frac{i}{\hbar} \int d\tau \sum_{\mathbf{k},\sigma} \left[L(\psi^*_{\mathbf{k}\sigma}, \psi_{\mathbf{k}\sigma}) + (\bar{\eta}_{\mathbf{k}\sigma}\psi_{\mathbf{k}\sigma} + \psi^*_{\mathbf{k}\sigma}\eta_{\mathbf{k}\sigma}) \right] \right\} \tag{61}$$

where the Lagrangian in this mixed "time-momentum" representation has the form

$$L(\psi^*_{k\sigma}, \psi_{k\sigma}) \;=\; \sum_{\mathbf{k},\sigma} \left[\hbar\psi^*_{\mathbf{k}\sigma}\partial_\tau\psi_{\mathbf{k}\sigma} \;-\; \mathcal{H}(\psi^*_{\mathbf{k}\sigma}, \partial_\tau\psi_{\mathbf{k}\sigma}) \right] \tag{62}$$

where $\mathcal{H}(\psi^*_{\mathbf{k}\sigma}, \psi_{\mathbf{k}\sigma})$ takes the same form as eq. (60), but with $\hat{c}^\dagger_{\mathbf{k},\sigma}$, $\hat{c}_{\mathbf{k},\sigma}$ now substituted with $\psi^*_{\mathbf{k}\sigma}$, $\psi_{\mathbf{k}\sigma}$. A point of notation here - the $\psi^*_{\mathbf{k}\sigma}$, $\psi_{\mathbf{k}\sigma}$ are Grassmann variables, NOT operators like the $\hat{c}^\dagger_{\mathbf{k},\sigma}$, $\hat{c}_{\mathbf{k},\sigma}$. We could just as easily have called them $\psi_{\mathbf{k},\sigma}$, $\bar{\psi}_{\mathbf{k},\sigma}$, so as to be consistent with $\eta_{\mathbf{k},\sigma}$, $\bar{\eta}_{\mathbf{k},\sigma}$; but we have not done this so as to distinguish the $\psi_{\mathbf{k}\sigma}$, $\psi^*_{\mathbf{k}\sigma}$ used here from the $\psi_{\mathbf{k},\sigma}$, $\bar{\psi}_{\mathbf{k},\sigma}$ used for relativistic Dirac fermions.

If we ignore the interactions in eqs. (60)-(62), we can go through the same kind of development that we already gave for the Dirac electron propagator, to find an expression for $Z_0[\bar{\eta}, \eta]$, and from there obtain the correlation functions for these free non-relativistic electrons. Thus, our FREE PARTICLE ACTION is

$$S_0 \;=\; \int d\tau \sum_{\mathbf{k},\sigma} \psi^*_{\mathbf{k}\sigma}(\tau) \left[\hbar\partial_\tau - \epsilon^0_{\mathbf{k}} \right] \psi_{\underline{k}\sigma}(\tau) \tag{63}$$

or, if we Fourier transform to frequency space,

$$S_0 \;=\; \int \frac{d\epsilon}{2\pi} \sum_{\mathbf{k},\sigma} \psi^*_{\mathbf{k}\sigma}(\epsilon) \left[\hbar\epsilon - \epsilon^o_{\mathbf{k}} \right] \psi_{\mathbf{k}\sigma}(\epsilon) \tag{64}$$

Then the normalized partition function for the system is

$$\mathcal{Z}_0[\bar{\eta}, \eta] \;=\; \frac{\int \mathcal{D}\,\psi^*_{\mathbf{k}\sigma} \int \mathcal{D}\psi_{\mathbf{k}\sigma}\, e^{\frac{i}{\hbar}\sum_{\mathbf{k},\sigma}\int \frac{d\epsilon}{2\pi} \left\{ \psi^*_{\mathbf{k}\sigma}(\epsilon)\left[\hbar\epsilon - \epsilon^o_{\mathbf{k}}\right]\psi_{\mathbf{k}\sigma}(\epsilon) \,+\, \left[\bar{\eta}_{\mathbf{k}\sigma}(\epsilon)\psi_{\mathbf{k}\sigma} \,+\, \psi^*_{\mathbf{k}\sigma}\eta_{\mathbf{k}\sigma}\right] \right\}}}{\int \mathcal{D}\,\psi^*_{\mathbf{k}\sigma} \int \mathcal{D}\psi_{\mathbf{k}\sigma}\, e^{\frac{i}{\hbar}\sum_{\mathbf{k},\sigma}\int \frac{d\epsilon}{2\pi} \psi^*_{\mathbf{k}\sigma}(\epsilon)\left[\hbar\epsilon - \epsilon^o_{\mathbf{k}}\right]\psi_{\mathbf{k}\sigma}(\epsilon)}} \tag{65}$$

and if we do the Gaussian integrals as before, we get, for $\mathcal{Z}_0[\bar{\eta}, \eta]$, that

$$\mathcal{Z}_0[\bar{\eta}, \eta] \;=\; \exp\left[\frac{i}{\hbar} \int \frac{d\epsilon}{2\pi} \sum_{\mathbf{k},\sigma} \bar{\eta}_{\mathbf{k}\sigma}(\epsilon) \left(\frac{1}{\hbar\epsilon - \epsilon^o_{\mathbf{k}}} \right) \eta_{\mathbf{k}\sigma}(\epsilon) \right] \tag{66}$$

Note the simplicity of this form compared to what we wrote for $\mathcal{Z}_0^{Dirac}$ in eq. (49); this is simply because we wrote eq. (49) in a position or time eigenbasis, in which it is non-diagonal;

it can easily be rewritten in an energy or momentum basis, when it will look like eq. (66). Since the $\eta_{\mathbf{k}\sigma}(\epsilon)$ are Grassmann variables, we can also write eq. (66) as the product:

$$\mathcal{Z}_0[\bar{\eta},\eta] = \prod_{\epsilon}\prod_{\mathbf{k},\sigma}\left[1 + \bar{\eta}_{\mathbf{k}\sigma}(\epsilon)\left(\frac{i/\hbar}{\hbar\epsilon - \epsilon^o_{\mathbf{k}}}\right)\eta_{\mathbf{k}\sigma}(\epsilon)\right] \tag{67}$$

where the products must be taken to the limit as $\mathbf{k}$ and ϵ become continuous.

The correlation functions must now be found. It is easy to show that

$$G_2(\mathbf{k},\epsilon) \quad = \quad \hbar^2 \frac{\delta^2 \mathcal{Z}_0[\bar{\eta},\eta]}{\delta\bar{\eta}_{\mathbf{k}\sigma}(\epsilon)\,\delta\eta_{\mathbf{k}\sigma}(\epsilon)} \quad = \quad \frac{i\hbar}{\hbar\epsilon - \epsilon^o_{\mathbf{k}}} \tag{68}$$

and so on, for the higher correlation functions.

NOTE ON THESE EXAMPLES: There are 2 important points not discussed here, regarding these examples. The first is that we did not include any interactions - we will rectify this in the next section. The second is that we assumed that the density of the fermions was negligible. But for fermions this is not usually realistic, at least in a many-body system. We deal with this properly when discussing finite temperatures. However the remedy here is simple; we make the substitution

$$\begin{aligned} &\epsilon^o_{\mathbf{k}} \;\rightarrow\; \epsilon^o_{\mathbf{k}} - \mu && \text{non}-\text{relativistic fermions} \\ &\left(|\mathbf{k}|^2 + m^2\right)^{\frac{1}{2}} \;\rightarrow\; \left(|\mathbf{k}|^2 + m^2\right)^{\frac{1}{2}} - \mu && \text{Dirac fermions} \end{aligned} \tag{69}$$