B1. The generating functional $\mathcal{Z}[J]$

The generating functional $\mathcal{Z}[J]$ is a key object in quantum field theory - as we shall see it reduces in ordinary quantum mechanics to a limiting form of the 1-particle propagator $G(x, x'|J)$ when the particle is under the influence of an external field $J(t)$. However, before beginning, it is useful to look at another closely related object, viz., the generating functional $\bar{Z}(J)$ in ordinary probability theory.

Recall that for a simple random variable ϕ, we can assign a probability distribution $P(\phi)$, so that the expectation value of any variable $A(\phi)$ that depends on ϕ is given by

$$\langle A \rangle = \int d\phi P(\phi) A(\phi), \tag{1}$$

where we have assumed the normalization condition

$$\int d\phi P(\phi) = 1. \tag{2}$$

Now let us consider the generating functional $\bar{Z}(J)$ defined by

$$\bar{Z}(J) = \int d\phi P(\phi) e^{J\phi}. \tag{3}$$

From this definition, it immediately follows that the "n-th moment" of the probability distribution $P(\phi)$ is

$$g_n \;=\; \langle \phi^n \rangle \;=\; \int d\phi P(\phi) \phi^n \;=\; \left. \frac{\partial^n \bar{Z}(J)}{\partial J^n} \right|_{J=0}, \tag{4}$$

and that we can expand $\bar{Z}(J)$ as

$$\begin{aligned} \bar{Z}(J) &= \sum_{n=0}^{\infty} \frac{J^n}{n!} \int d\phi P(\phi) \phi^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\, g_n J^n, \end{aligned} \tag{5}$$

so that $\bar{Z}(J)$ acts as a "generator" for the infinite sequence of moments g_n of the probability distribution $P(\phi)$.

For future reference it is also useful to recall how one may also expand the logarithm of $\bar{Z}(J)$; we write,

$$\begin{aligned} \bar{Z}(J) &= \exp[\bar{W}(J)] \\ \bar{W}(J) &= \ln \bar{Z}(J) = \ln \int d\phi P(\phi) e^{J\phi}. \end{aligned} \tag{6}$$

Now suppose we make a power series expansion of $\bar{W}(J)$, i.e.,

$$\bar{W}(J) = \sum_{n=0}^{\infty} \frac{1}{n!}\, C_n J^n, \tag{7}$$

where the C_n, known as "cumulants", are just

$$C_n \;=\; \left.\frac{\partial^n \bar{W}(J)}{\partial J^n}\right|_{J=0}. \tag{8}$$

The relationship between the cumulants C_n and moments g_n is easily found. We notice that

$$\begin{aligned}
\frac{\partial \bar{W}(J)}{\partial J} &= \frac{\partial \ln \bar{Z}(J)}{\partial J} &&= \frac{1}{\bar{Z}}\frac{\partial \bar{Z}(J)}{\partial J} \\
\frac{\partial^2 \bar{W}(J)}{\partial J^2} &= \frac{\partial}{\partial J}\left(\frac{1}{\bar{Z}}\frac{\partial \bar{Z}(J)}{\partial J}\right) &&= \left[\frac{1}{\bar{Z}}\frac{\partial^2 \bar{Z}(J)}{\partial J^2} - \frac{1}{\bar{Z}^2}\left(\frac{\partial \bar{Z}(J)}{\partial J}\right)^2\right],
\end{aligned} \tag{9}$$

and so on. Continuing in this way, and using eq.(6), we can find the $\{C_j\}$ in terms of $\{g_j\}$, by expanding $\ln \bar{Z}[J] = \ln\left\{\sum_n \frac{1}{n!} g_n J^n\right\}$, and comparing coefficients of J^n. We then find that

$$\begin{aligned}
C_1 &= g_1, \\
C_2 &= g_2 - g_1^2, \\
C_3 &= g_3 - 3g_1 g_2 + 2g_1^3, \\
C_4 &= g_4 - 3g_2^2 - 4g_1 g_3 + 12 g_2 g_1^2 - 6g_1^4;
\end{aligned} \tag{10}$$

and so on.

Finally, note that in simple probability theory, we can make a "Legendre" transformation to change the dependent variables; if we write

$$\bar{W}(J) = \bar{\Gamma}(\phi) + J\phi, \tag{11}$$

then in terms of the new function $\bar{\Gamma}(\phi)$, we have

$$\frac{\partial \bar{\Gamma}(\phi)}{\partial \phi} = -J. \tag{12}$$

and we can then rewrite everything in terms of $\bar{\Gamma}(\phi)$ and its derivatives.

We will see in what follows that all of these functions of simple probability theory can be put into correspondence with functionals in QFT. As we will also see, it is useful and illuminating to explore the analogies between probability theory, field theory, and thermodynamics/ statistics mechanics.

B.1.1: GENERATING FUNCTIONAL and PROPAGATORS for SCALAR FIELD THEORY

Suppose we are now dealing with a quantum field $\phi(x)$ in D dimensions. Then the analogue of the generating function in eq.(3) for ordinary probability distribution is a generating *functional* $\mathcal{Z}[J]$, given by

$$\begin{aligned}\mathcal{Z}[J] &= N\int D\phi\; A(\phi; J=0)\; e^{i\int d^Dx J(x)\phi(x)} \\ &= N\int D\phi\; e^{\frac{i}{\hbar}S[\phi]}\; e^{i\int d^Dx J(x)\phi(x)},\end{aligned} \tag{13}$$

and we see that the analogue of the probability distribution $P(\phi)$ is the "partition function" $\mathcal{Z}[J=0,\phi]$, i.e., the normalization factor N in eq. (13) is

$$\begin{aligned}N^{-1} &= \mathcal{Z}[J=0] = \int D\phi e^{i/\hbar S[\phi]} \;\equiv\; \int D\phi A(\phi), \\ A(\phi) &= e^{i/\hbar S[\phi]} \;=\; e^{i/\hbar\int d^Dx L(\phi(x))},\end{aligned} \tag{14}$$

so that the analogue of the probability distribution $P(\phi)$ for a random variable to take the value ϕ is now the *amplitude* $A(\phi) = e^{i/\hbar S[\phi]}$ for a field to take a given configuration $\phi(x)$.

We shall see later what we must write to deal with probabilities (as opposed to amplitudes) in QFT, using a kind of "double path integral" technique (recall that in ordinary QM one discusses probabilities using the density matrix, which also can be understood in terms of a double path integral).

Note also that a convergence factor has been left out of eqs.(13) and (14). Just as in ordinary QM, we need to specify which configurations are being integrated over, and with which boundary conditions. In ordinary QM, this is done by rotating the end points of the time integration to $\pm i\infty$, or by adding a term to the exponent to force the Green function $\int Dq e^{i/\hbar S[q]}$ to pick out the ground state (see notes on path integration for QM, in section A). The same is done here - we add a convergence factor, to get

$$\begin{aligned}\mathcal{Z}[J] &= N\int D\phi\; A(\phi; J=0)\; e^{i/\hbar\int d^Dx[J(x)\phi(x)+i\epsilon\phi^2(x)]} \\ &= N\int D\phi\; e^{i/\hbar\int d^Dx[L(\phi(x))+J(x)\phi(x)+i\epsilon\phi^2(x)]}.\end{aligned} \tag{15}$$

I do not go through the demonstration that this gives the desired result, since it is similar to (but more complicated than) the demonstration for ordinary Q.M.. In any case, the key point here is that we require that

$$\mathcal{Z}[J] = \langle 0|0\rangle_{J(x)}, \tag{16}$$

i.e., that $\mathcal{Z}[J]$ is the vacuum expectation value for the theory, under the influence of an external source $J(x)$ (NB: this then fixes the normalization N in (15)).

From now on all reference to the convergence factor in eq.(15) is suppressed. However, I will introduce a special symbol to indicate that we are using a specially chosen contour to define the vacuum state: in future I will write "$\oint \mathcal{D}\phi$" to denote a functional integral over a field $\phi(x)$ which singles out the vacuum state. This symbol acknowledges what I am not showing here, viz., that one can also define this vacuum using a kind of "closed path" integration, first discussed in various forms by Schwinger, DeWitt, and Keldysh back in the early 1960's (this is discussed in detail in the Appendices).

Continuing now in analogy with ordinary Q.M., and with the simple theory of probability, let us define the time-ordered correlators

$$G_n(x_1, \cdots, x_n) \;=\; \langle 0|\hat{T}\{\phi(x_1), \cdots, \phi(x_n)\}|0\rangle \;=\; \frac{\oint D\phi\; e^{i/\hbar S[\phi]}\; \phi(x_1)\cdots\phi(x_n)}{\oint D\phi\; e^{i/\hbar S[\phi]}}, \tag{17}$$

where the denominator is added to normalize the result (we cannot in general assume the analogue of eq.(2)); the corresponding power series expansion is then (cf. eq.(5))

$$\mathcal{Z}[J] \;=\; \mathcal{Z}[0] \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int dx_1 \cdots \int dx_n\; G_n(x_1, \cdots, x_n)\; J(x_1)J(x_2)\cdots J(x_n) \tag{18}$$

and the G_n is related to $\mathcal{Z}[J]$ by

$$G_n(x_1, \cdots, x_n) \;=\; (-ih)^n \frac{1}{\mathcal{Z}[0]}\; \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0} \tag{19}$$

(cf. eq.(4)).

Let us now note that the factor $\mathcal{Z}[0]$ is actually unity, ie., $\mathcal{Z}[0] = 1$; this is because it represents that amplitude for the vacuum state to remain the vacuum state over the time period from $t = -\infty$ to $t = \infty$, when no external current $J(x)$ is being applied, ie., $\mathcal{Z}[0] = \langle 0|0\rangle$. Since the system cannot then be excited, it must remain in the vacuum state. Thus henceforth we will drop the factor $\mathcal{Z}[0]$.

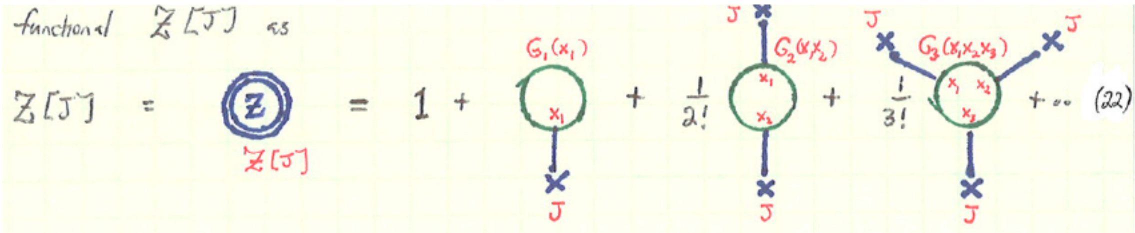
Noting that $Z(J)$ in ordinary probability theory is just the expectation value of the function $e^{\phi J}$, we see that in scalar field theory, we have

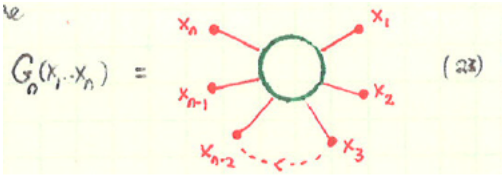
$$\mathcal{Z}[J] \;=\; \langle 0|\hat{T}\{e^{i/\hbar \int dx J(x)\phi(x)}\}|0\rangle \tag{20}$$

which when expanded out gives

$$\begin{aligned}\mathcal{Z}[J] \;=\; \langle 0|0\rangle \;&+\; \frac{i}{\hbar}\int d^D x \langle 0|\phi(x)|0\rangle J(x) \\ &+\; \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2 \int d^D x_1 \int d^D x_2\; \langle 0|\hat{T}\{\phi(x_1)\phi(x_2)\}|0\rangle\; J(x_1)J(x_2) \;+\ldots\end{aligned} \tag{21}$$

in agreement with eqs.(15) and (16).





It is useful to represent the formal results diagrammatically. We represent the vacuum functional $Z[J]$ as the diagrammatic eqtn. (22) which is just the series sum in eqtns. (18) and (21); here we assume the Green functions are represented diagrammatically as the diagrammatic eqtn. (23)

These diagrams are drawn according to well-defined rules, which can be read off from the relevant equations. Thus we have the correspondences:

$$f(x) \longrightarrow \times J \quad \equiv \quad \frac{i}{\hbar}\int d^D x J(x) f(x) \tag{24}$$

where the cross indicates the external source current $J(x)$, and this is connected by the line, or "external leg", to some function $f(x)$. Thus we multiply whatever this "external leg" is attached to, by the factor $\frac{i}{\hbar}J(x)$, and then integrate the combination over x. This is clear in eqs. (18) and (21). Note that whereas $G_n(x_1, \cdots, x_n)$ has n external legs, $\mathcal{Z}[J]$ has none - they are all integrated over.

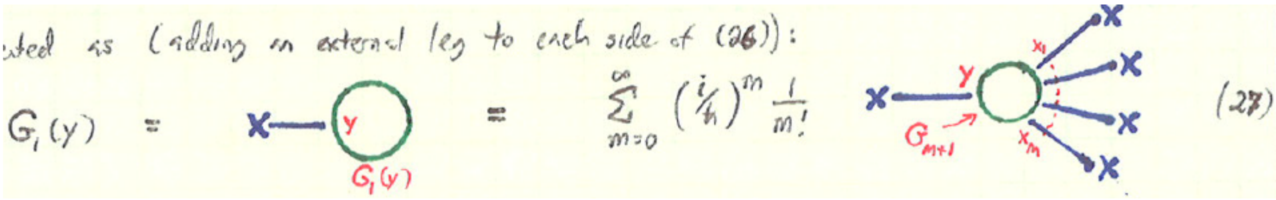
We get another useful relation, of recursive form, between $G_n(y)$ and the generating function $\mathcal{Z}[J]$ by differentiating eq. (20), and expanding the right hand side; we have $G_1(y) = -i\hbar\, \delta\mathcal{Z}[J]/\delta J(y)|_{J=0}$, so that

$$G_1(y) \quad = \quad -i\hbar\frac{\delta}{\delta J(y)}\sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{1}{n!}\int dx_1 \ldots dx_n G_n(x_1, \cdots, x_n)\; J(x_1)\ldots J(x_n)\bigg|_{J=0} \tag{25}$$

and reorganizing the labelling in the series, we get

$$G_1(y) \quad = \quad \lim \sum_{m=0}^{\infty}\left(\frac{i}{\hbar}\right)^m \frac{1}{m!}\int dx_1 \ldots dx_m\; G_{m+1}(x_1, \cdots, x_m; y)\; J(x_1)\ldots J(x_m)\bigg|_{J=0} \tag{26}$$

which is represented diagrammatically in (27) (adding an external leg to each side of eq. (26)):



Note that this corresponds to a rather trivial result in ordinary probability theory; from eq. (5) we have

$$g_1 = \langle\phi\rangle \;=\; \int d\phi\; P(\phi)\;\phi\; e^{\phi J}|_{J=0} \;=\; \sum_{n=0}^{\infty}\frac{J^n}{n!}\int d\phi\;\phi^{n+1}P(\phi)|_{J=0} \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\,g_{n+1}J^n|_{J=0} \tag{28}$$

in analogy to (26).

So far everything we have said has been true for the Lagrangian density $\mathcal{L}[\phi(x)]$, i.e., we have not used a specific interaction like a $\phi^4(x)$ term. This will come later - first we have to develop more of the formal theory.

B.1.2: CONNECTED GREEN FUNCTIONS for SCALAR FIELD THEORY

We have seen how the generating functional or partition function or vacuum expectation value $\mathcal{Z}[J]$ corresponds to the generating function $\bar{Z}(J)$ for simple probability distributions. Let us now continue this analogy, and write

$$\mathcal{Z}[J] \;=\; e^{\frac{i}{\hbar}W[J]} \tag{29}$$

(compare eq. (6)). We now proceed in complete analogy with eqs. (6)-(10), by defining what we call "connected" Green functions (the name will become clear) as

$$\mathcal{G}_n^{(c)}(x_1,\cdots,x_n) \;=\; (-ih)^n\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}|_{J=0} \tag{30}$$

and

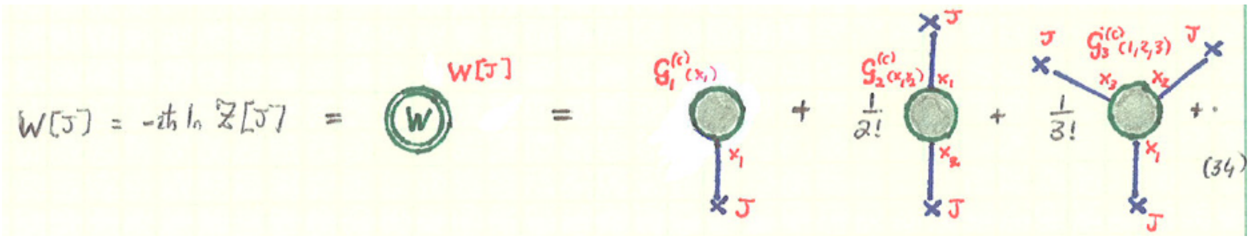
$$W[J] \;=\; -i\hbar\sum_{n=1}^{\infty}\left(\frac{i}{\hbar}\right)^n\frac{1}{n!}\int dx_1\ldots dx_n\,\mathcal{G}_n^{(c)}(x_1,\cdots,x_n)\,J(x_1)\ldots J(x_n) \tag{31}$$

and we also have

$$\frac{1}{\mathcal{Z}[J]}\frac{\delta\mathcal{Z}[J]}{\delta J} \;=\; \frac{i}{\hbar}\frac{\delta\bar{W}[J]}{\delta J} \tag{32}$$

Actually the closer analogy between $\mathcal{Z}[J]$ and probability theory is with the "characteristic function" $Z(J)$ (as opposed to $\bar{Z}(J)$ in eqs. (1)-(9)), defined by

$$Z(J) \;=\; \int d\phi P(\phi)e^{iJ\phi} \;=\; \langle e^{iJ\phi}\rangle \tag{33}$$



which can also be thought of as a simple Fourier transform of $P(\phi)$. To differentiate between all of these, we have put a bar over quantities like $\bar{W}$ or $\bar{Z}$ to indicate functions that are the results of simple exponentiation (like $\bar{Z}(J) = e^{\bar{W}(J)}$) as opposed to Fourier transformation (like $Z(J) = e^{\frac{i}{\hbar}W(J)}$).

The connected Green functions here are the analogues of the "cumulants" we defined for the ordinary probability distributions. There are however differences between the results of ordinary probability theory, and what we do here - the 2 main ones being:

(a) we now use $W[J] = -i\hbar \ln \mathcal{Z}[J]$ instead of $\bar{W}(J) = \ln Z[J]$, since we deal with the quantum mechanics (see footnote [1] below); and

(b) we now deal with functionals instead of functions.

One can also develop a graphical representation of the relations governing the connected Green functions. Notice first of all from eq. (31) that we can make exactly the same sort of graphic expansion for $W[J]$ as we did for $\mathcal{Z}[J]$; this is shown in the figure in (34). Here the connected Green functions are shown as solid circles. As before we can also write an eqtn. relating the functional derivative of $W[J]$, ie., $\mathcal{G}_1^{(c)}(y)$, to the series expansion of $W[J]$ (cf. eqs. (25) and (26)); since $\mathcal{G}_1^{(c)}(y) = -i\hbar\, \delta\bar{W}[J]/\delta J(y)|_{J=0}$ (compare eqtn. (32)), we have

$$\mathcal{G}_1^{(c)}(y) \;=\; \sum_{m=1}^{\infty} \left(\frac{i}{\hbar}\right)^m \frac{1}{m!} \int dx_1 \ldots dx_m\, \mathcal{G}_{m+1}^{(c)}(x_1, \cdots, x_m; y) J(x_1) \ldots J(x_m)\Big|_{J=0} \qquad (35)$$

which can be shown diagramatically in the same way as in (27).

We would also like to know how to explicitly relate the full correlators $G_n(x_1 \cdots, x_n)$ to the connected correlators $\mathcal{G}_n^{(c)}(x_1, \cdots, x_n)$. This is a little more messy than the analogous

[1]**NOTE:** It is common in the literature to write $\mathcal{Z}[J] = e^{\bar{W}[J]}$, in very close analogy with eq. (6). We then have the definitions and results as follows:

$$\begin{aligned}
\bar{W}[J] &= \sum_{n=0}^{\infty} \frac{1}{n!}\, \bar{\mathcal{G}}_n^{(c)}(x_1, \ldots, x_n)\, J(x_1) \ldots J(x_n) \\
\bar{\mathcal{G}}_n^{(c)}(x_1, \ldots, x_n) &= \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)}\Big|_{J=0} \\
\frac{1}{\mathcal{Z}[J]} \frac{\delta \mathcal{Z}[J]}{\delta J} &= \frac{\delta \bar{W}[J]}{\delta J} \\
G_1(y) &= -i\hbar \frac{\delta \mathcal{Z}[J]}{\delta J(y)} \;=\; Z[J]\, \mathcal{G}_1^{(c)}(y) \;=\; -i\hbar \mathcal{Z}[J]\, \bar{\mathcal{G}}_1^{(c)}(y)
\end{aligned}$$

relation between correlators and cumulants in eqtn. (10). Following our previous manoeuvres (compare eqtn. (9)) we repeatedly functionally differentiate (29) with respect to $J(x)$ and then let $J(x) \to 0$. Then, eg., at 2nd order we get

$$G_2(x_1, x_2) \;=\; \mathcal{G}_2^{(c)}(x_1, x_2) \;+\; \mathcal{G}_1^{(c)}(x_1)\mathcal{G}_1^{(c)}(x_2) \tag{36}$$

and if we continue this process, and take all possible internal permutations of vertices, we get

$$G_n(x_1, \cdots, x_n) \;=\; \sum_{\hat{P}:\{m_j\}}^{\sum_j m_j = n} \mathcal{G}_{m_1}^{(c)}(x_1 \cdots, x_{m_1}) \cdots\cdots \mathcal{G}_{m_n}^{(c)}(x_{n-m_n} \cdots, x_n) \tag{37}$$

where $\hat{P} : \{m_j\}$ denoted the set of all permutations of the $\{m_j\}$; thus, the total propagator is the sum of all possible way of adding together combinations of disconnected pieces of the connected propagators.

If this argument seems too glib, a more rigorous algebraic argument is given as the end of this chapter, where the exponential in eq. (29) is written explicitly as a sum of products over connected graphs.

B.1.3 VERTEX FUNCTIONS for SCALAR FIELD THEORY

So far we have expanded two different functionals as infinite series, i.e., the vacuum energy generating functional $\mathcal{Z}[J]$ (the partition function), and the generating functional $W[J]$ for connected diagrams (analogous to the free energy in statistics mechanics), related as before by $W[J] = -i\hbar \ln \mathcal{Z}[J]$, so that as we already saw,

$$\begin{aligned} \mathcal{Z}[J] \;&=\; \sum_n \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \prod_{j=1}^n \int d^D x_j \; G_n(x_1, \ldots, x_n) \; J(x_j) \\ \frac{i}{\hbar} W[J] \;&=\; \sum_n \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \prod_{j=1}^n \int d^D x_j \; \mathcal{G}_n^{(c)}(x_1, \ldots, x_n) \; J(x_j) \end{aligned} \tag{38}$$

Now we introduce a 3rd expansion, in terms of a new generating functional $\Gamma[\phi]$; this however, is not a functional of the external currents, but of the field $\phi(x)$ itself. We write

$$\Gamma[\phi] \;=\; \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots \int d^4x_n \; \Gamma_n(x_1, \ldots, x_n) \; \phi(x_1) \ldots \phi(x_n) \tag{39}$$

so that

$$\Gamma_n(x_1, \ldots, x_n) \;=\; \left. \frac{\delta^n \Gamma[J]}{\delta\phi(x_1) \cdots \delta\phi(x_n)} \right|_{\phi=0} \tag{40}$$

Now of course the way to change variables between the $J(x_i)$ and the $\phi(x_i)$ is by a Legendre transformation, with form given here by

$$\Gamma[\phi] \; = \; W[J] - \int d^4x J(x)\phi(x) \tag{41}$$

Thus now we now have the pair of relations

$$\begin{aligned} \left.\frac{\delta W[J]}{\delta J(x)}\right|_{\phi} &= \phi(x) \\ \left.\frac{\delta \Gamma[J]}{\delta \phi(x)}\right|_{J} &= -J(x) \end{aligned} \tag{42}$$

Later on we will look in some detail at vertex functions like $\Gamma_n(x_1, \ldots, x_n)$, which turn out to be central to the approach of QFT to the dynamics of real physical systems - they are used in everything from scattering theory to particle physics to transport theory in condensed matter systems.

Here we will simply establish the relationship between the connected Green functions $\mathcal{G}_n^{(c)}(x_1, \ldots, x_n)$ and the proper vertices $\Gamma_n(x_1, \ldots, x_n)$. Let's first do this for the simplest case, that of the 2nd-order correlators $\mathcal{G}_2^{(c)}(x_1, x_2)$ and $\Gamma_2(x_1, x_2)$. From their definitions, and from eq. (42), we have

$$\begin{aligned} \mathcal{G}_2^{(c)}(x_1, x_2) &= -\hbar^2 \left.\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} = -\hbar^2 \left.\frac{\delta \phi(x_1)}{\delta J(x_2)}\right|_{J=0} \\ \Gamma_2(x_1, x_2) &= \left.\frac{\delta^2 \Gamma[J]}{\delta \phi(x_1)\delta \phi(x_2)}\right|_{\phi=0} = -\left.\frac{\delta J(x_1)}{\delta \phi(x_2)}\right|_{\phi=0} \end{aligned} \tag{43}$$

from which we immediately find that

$$\int d^4y \; \mathcal{G}_2^{(c)}(x_1, y) \; \Gamma_2(y, x_2) \; = \; \frac{\delta \phi(x_1)}{\delta \phi(x_2)} \; = \; \hbar^2 \delta(x_1 - x_2) \tag{44}$$

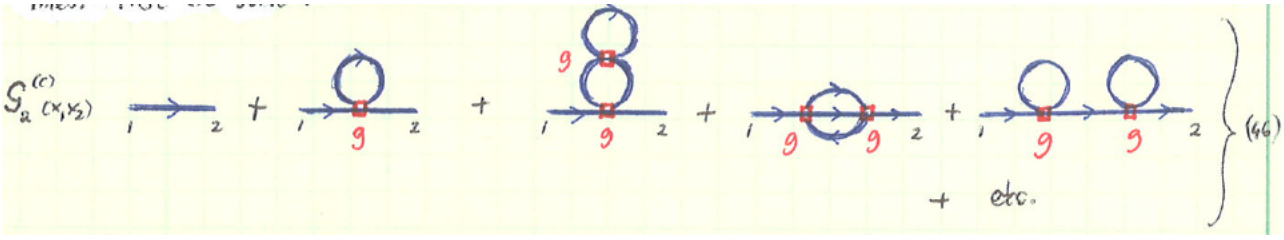
whose Fourier transform is

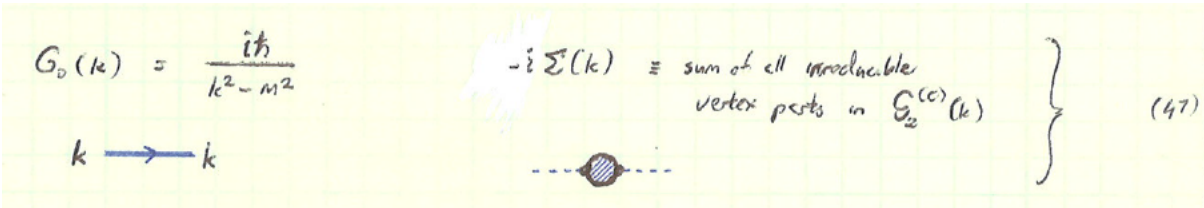
$$\mathcal{G}_2^{(c)}(p, -p) \; = \; \frac{\hbar^2}{\Gamma_2(p, -p)} \tag{45}$$

i.e., the two functions are inverses of each other.

To understand this result a little better it is helpful to look at it diagrammatically. Let's consider what kind of graphs make up $\mathcal{G}_2^{(c)}(x_1, x_2)$. As we see above, the $\mathcal{G}_n^{(c)}(x_1, \ldots, x_n)$ are "connected graphs", which cannot be separated without cutting internal lines. Here we show some in eqtn. (46).

We now notice that we can divide graphs like those above into 2 kinds. There are those graphs which, apart from their ingoing and outgoing lines, cannot be separated into 2 parts by cutting any internal lines - or to put it differently, they can be viewed as "internal irreducible " parts, connected to 2 external legs. Note that all of these "irreducible vertex





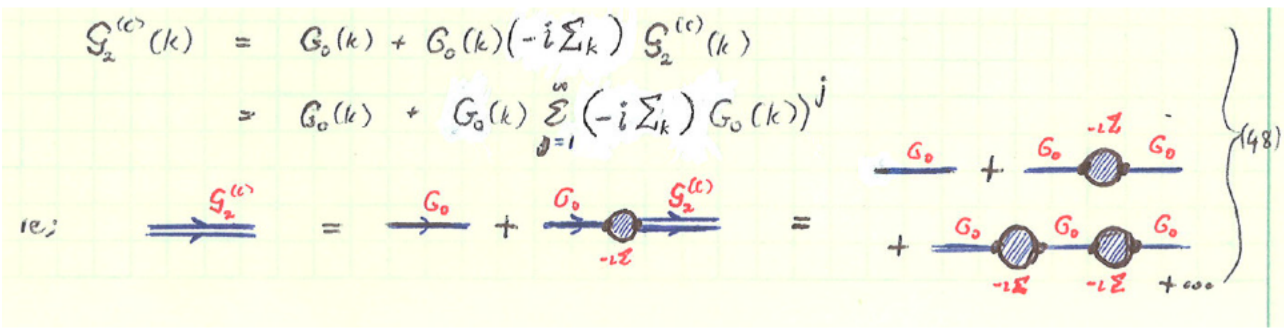
parts" are different from each other. Then we have graphs that can be separated into 2 parts by cutting an internal line; these are called "reducible" graphs. We see that we can get all graphs for $\mathcal{G}_2^{(c)}(x_1, x_2)$ by simply stringing together the irreducible parts in all possible ways.

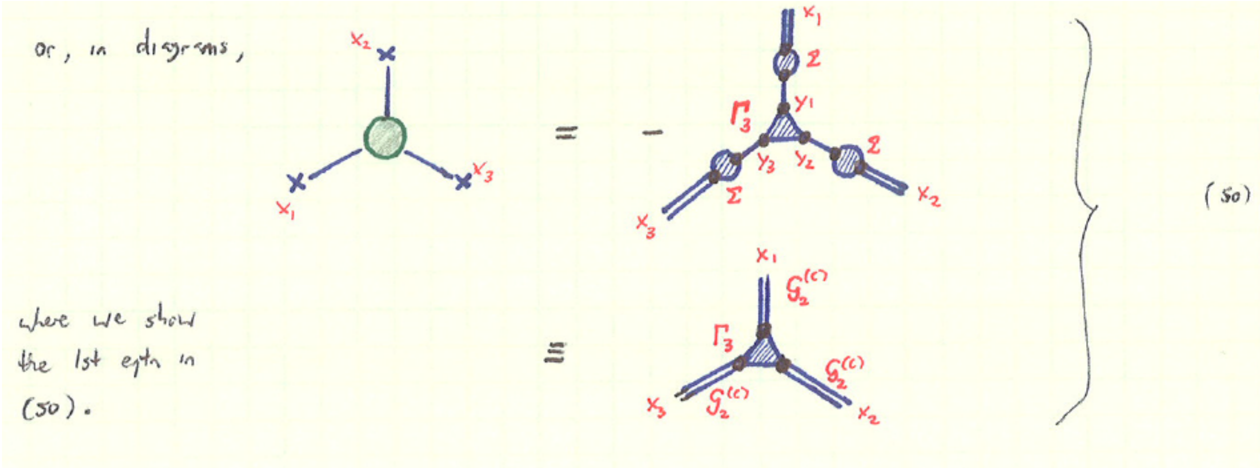
Let us put this formally; we define the following 2 quantities shown in (47), where $G_0(k)$ is of course just the free propagator, which we have already met in our discussion of 1-particle Green functions in QM. The diagram for $-i\sum(k)$ has no external legs - the dashed lines simply indicate where they would attach to $-i\sum(k)$. We work here in 4-momentum space, simply because momentum is conserved. Now we obtain the full $\mathcal{G}_2^{(c)}(k) \equiv \mathcal{G}_2^{(c)}(k, -k)$ as (48) and so we have

$$
\begin{aligned}
\mathcal{G}_2^{(c)}(k) &= \frac{i\hbar}{k^2 - m^2 - \hbar\Sigma(k)} \\
\Gamma_2(k) &= -i\hbar\left[k^2 - m^2 - \hbar\Sigma(k)\right]
\end{aligned} \tag{49}
$$

and we see that the vertex part, apart from the term $(k^2 - m^2)$ can be identified with $-i\sum(k)$, which is of course just the self-energy.

One can continue this analysis for the higher-order functions $\mathcal{G}_n^{(c)}(x_1, \ldots, x_n)$ and $\Gamma_n(x_1, \ldots, x_n)$.





Without going through the derivation, we find that

$$\begin{aligned}
\mathcal{G}_3^{(c)}(x_1, x_2, x_3) &= -\int d^4y_1 \int d^4y_2 \int d^4y_3\, \Gamma_3(y_1, y_2, y_3)\, \mathcal{G}_2^{(c)}(x_1, y_1)\mathcal{G}_2^{(c)}(x_2, y_2)\mathcal{G}_2^{(c)}(x_3, y_3) \\
\Gamma_3^{(c)}(x_1, x_2, x_3) &= -\int d^4y_1 \int d^4y_2 \int d^4y_3\, \mathcal{G}_3^{(c)}(y_1, y_2, y_3)\, \Gamma_2^{(c)}(x_1, y_1)\Gamma_2^{(c)}(x_2, y_2)\Gamma_2^{(c)}(x_3, y_3)
\end{aligned} \qquad (50)$$

as shown in the diagrams of (50). One can continue to Γ_4, which can be written in terms of $\mathcal{G}_4^{(c)}, \mathcal{G}_3^{(c)}, \mathcal{G}_2^{(c)}$, Γ_3 and Γ_2; and so on; all the $\Gamma_n(x_1, \ldots, x_n)$ have the same "tree" structure.

We have now written expressions for $\mathcal{Z}[J]$ in terms of 3 different sets of quantities, i.e., the correlators $G_n(x_1, \ldots, x_n)$, the connected correlators $\mathcal{G}_n^{(c)}(x_1, \ldots, x_n)$, and the proper vertices $\Gamma_n(x_1, \ldots, x_n)$. Now it turns out that each of these has an important physical meaning, but to better appreciate that we have to first go on with our formal developments.

B.1.4: FREE PROPAGATORS for ϕ-FIELD

Before we ever look at the effect of interactions on the field dynamics (and this is of course the whole point of QFT), we need to establish the properties of the free field, without interactions. What this means, as far as we are concerned, is determining the form of the correlation functions. Thus we will begin again with the generating functional $\mathcal{Z}_0[J]$ for the ϕ-field scalar theory, given in eqs. (13) and (15) above. Let us write this out explicitly for the free system, i.e.,

$$\mathcal{Z}_0[J] = \frac{\int D\phi\, e^{\frac{i}{\hbar}\int d^Dx[L_0(\phi)+\phi(x)J(x)+i\delta\phi^2(x)]}}{\int D\phi\, e^{\frac{i}{\hbar}\int d^Dx[L_0(\phi)+i\delta\phi^2(x)]}} \qquad (51)$$

where we include the convergence factor $i\delta\phi^2(x)$ in the exponent, so as not to forget it. Now actually we can evaluate eq. (51) exactly because the functional integrals involve quadratic forms. To make this more explicit, let's rewrite the free field action, noting that (NB: $\partial^2 = \partial^\mu\partial_\mu$ is often written as $\Box$ or Δ):

$$\int d^D x\,(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2) \;=\; \int d^D x\left[\partial_\mu(\phi\partial^\mu\phi) - \phi\partial^2\phi - m^2\phi^2\right]$$
$$= \oint_S d^{D-1}x\,(\phi\partial^\mu\phi) \;-\; \int d^D x\,\phi(\partial^2+m^2)\phi \tag{52}$$

using the D-dimensional variant of Gaussian theorem; here $\oint_S d^{D-1}x$ denotes a surface integral. Thus we have the alternative form for $\mathcal{Z}_0[J]$ as

$$Z_0[J] \;=\; \frac{\int D\phi\; e^{-iQ_J^0(\phi)/\hbar}}{\int D\phi\; e^{-iQ_0(\phi)/\hbar}} \tag{53}$$

where we drop the total derivative "surface integral" term in eq. (52), simply assuming that $|\phi| \to 0$ fast enough at the boundaries of whatever system we deal with; and where we define

$$Q_0(\phi) \;=\; \frac{1}{2}\int d^D x\;\phi(x)\left[(\partial^2+m^2)\right]\phi(x) \;\equiv\; (\phi, Q_0\phi)$$
$$Q_J^0(\phi) \;=\; \int d^D x\left\{\frac{1}{2}\phi(x)\left[(\partial^2+m^2)\right]\phi(x) - J(x)\phi(x)\right\} \;\equiv\; (\phi, Q_J^0\phi) \tag{54}$$

These are standard Gaussian functional integrals, and we immediately get

$$\int D\phi\; e^{-iQ_0(\phi)/\hbar} \;=\; |\det iQ_0/\hbar|^{-\frac{1}{2}}$$
$$\int D\phi\; e^{-iQ_J^0(\phi)/\hbar} \;=\; |\det iQ_0/\hbar|^{-\frac{1}{2}}\, e^{\frac{i}{2\hbar}(J,Q_0^{-1}J)} \tag{55}$$

where $Q_0 = (\partial^2 + m^2)$. The determinants cancel in eq. (53), and we are left with

$$\mathcal{Z}_0[J] \;=\; e^{-\frac{i}{2\hbar}\int d^D x \int d^D x' J(x)\Delta_F(x-x')J(x')} \tag{56}$$

where the "Feynman propagator" is

$$\Delta_F(x) \;=\; -(\partial^2 + m^2 - i\delta)^{-1} \tag{57}$$

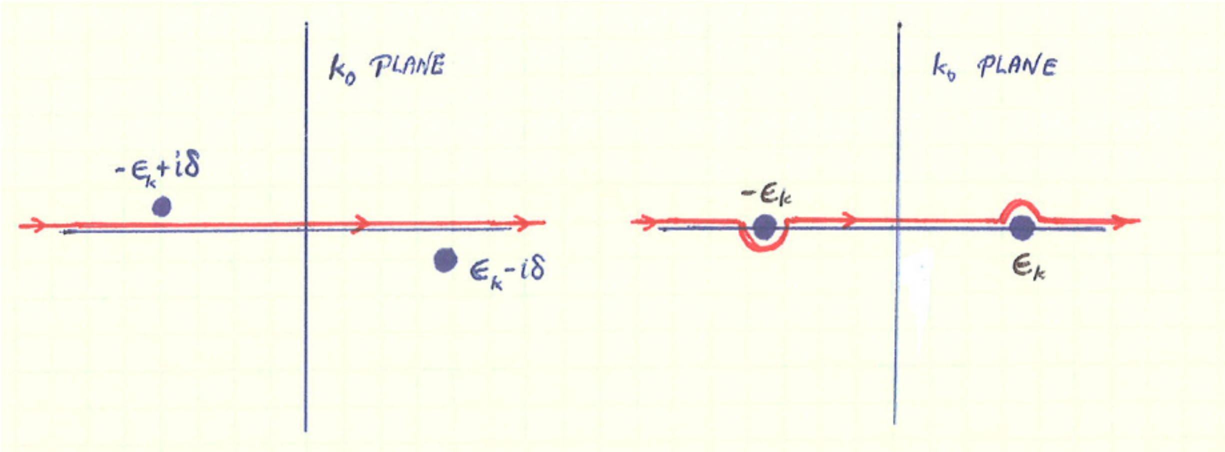
i.e., it is defined by

$$(\partial^2 + m^2 - i\delta)\;\Delta_F(x) \;=\; -\delta^4(x) \tag{58}$$

where $\delta^D(x)$ is the D-dimensional δ-function. Thus we have reduced the generating functional $\mathcal{Z}_0[J]$ to a simple function involving integrals over $J(x)$ and $\Delta_F(x-x')$.

The Feynman propagator $\Delta_F(x-x')$ is the object that ties together the currents $J(x)$ and $J(x')$, and it is therefore important to understand its properties. Since the free field system is translationally invariant, we Fourier transform it to get

$$\Delta_F(x) \;=\; \int \frac{d^4k}{(2\pi)^4}\Delta_F(k)\; e^{-ikx} \;=\; \sum_k e^{-ikx}\Delta_F(k) \tag{59}$$



where

$$\Delta_F(k) \;=\; \frac{1}{k^2 - m^2 + i\delta} \;=\; \frac{1}{k_0^2 - (|\mathbf{k}|^2 + m^2) + i\delta} \tag{60}$$

where the last form we use the Lorentz invariance of the theory. From this last form we can see the pole structure of the theory in the frequency domain (i.e., in k_0-space).

Let's see what this means when we come to integrate over k-space, as defined by eq. (59). If we do the frequency integral first, then we must follow the path shown in the left of the figure - to see this, just rewrite eq. (60) as

$$\Delta_F(k) \;=\; \frac{1}{2k_0}\left[\frac{1}{k_0 - (\epsilon_{\mathbf{k}} - i\delta)} + \frac{1}{k_0 + (\epsilon_{\mathbf{k}} + i\delta)}\right] \tag{61}$$

where

$$\epsilon_{\mathbf{k}}^2 \;=\; (|\mathbf{k}|^2 + m^2) \tag{62}$$

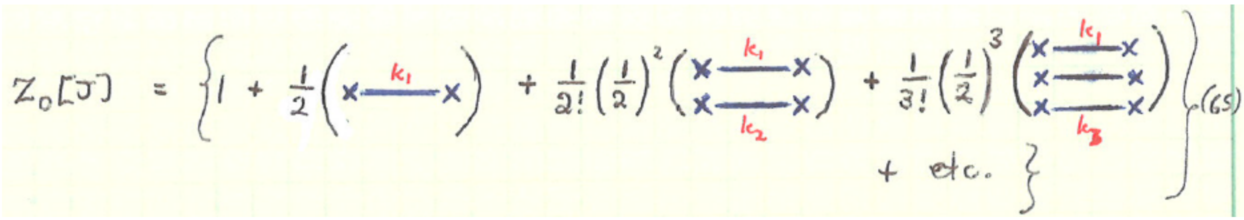
The contour can also be taken as shown in the right of the figure, if we desire to put the poles at $\pm(\epsilon_k - i\delta)$ onto the real axis instead (ie., let $\delta = 0$). Note that in both cases we are simply using artificial devices to deal with the branch cut structure of the propagator - something we have already seen in ordinary QM (in section A).

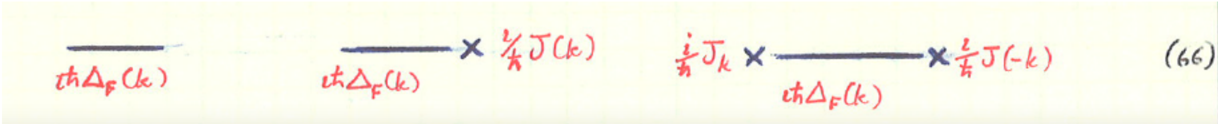
Let's now look at what we get for the correlation functions in this free field theory. The simplest thing to do is to simply expand the function in eq. (56), to get

$$\mathcal{Z}_0[J] \;=\; 1 \;+\; \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{-i}{2\hbar}\right)^n \left[\int d^Dx \int d^Dx' J(x)\Delta_F(x-x')J(x')\right]^n \tag{63}$$

a series which is obviously better written in terms of the Fourier transformation of $J(x)$ and $\Delta_F(x)$; it is easy to show that we get

$$\mathcal{Z}_0[J] \;=\; 1 \;+\; \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{-i}{2\hbar}\right)^n \left[\sum_k J_k\, \Delta_F(k)\, J_{-k}\right]^n \tag{64}$$





and we see that both of these expressions have obvious diagrammatic versions; we show this in (65), where the notation is pretty much as used in eq. (24); we have the diagram in (66), with integration over repeated indices (in this case the momenta).

To obtain the correlation function we go back to the exponential form in eq. (56), and just functionally differentiate. Let's do the function $G_2(x,x')$ in detail. From eq. (19) we have

$$G_2^{(0)}(x,x') \;=\; -\hbar^2 \frac{\delta^2 \mathcal{Z}_0[J]}{\delta J(x)\delta J(x')}|_{J=0} \tag{67}$$

Now, using the simple result

$$\frac{\delta}{\delta J(x)} e^{\frac{-i}{2\hbar}\int dx_1 \int dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)} \;=\; \frac{-i}{\hbar}\int dx_1 \Delta_F(x_1-x_2)J(x_1)\; e^{\frac{-i}{2\hbar}\iint J\Delta_F J} \tag{68}$$

We easily find that

$$G_2^{(0)}(x,x') \;=\; i\hbar\Delta_F(x_1-x_2) \tag{69}$$

By continuing this analysis we easily find that $G_3^{(0)}(x_1,x_2,x_3)=0$, and that

$$\begin{aligned} G_4^{(0)}(x_1,..,x_4) \;=\; & -\hbar^2\,[\Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\ & + \;\Delta_F(x_1-x_4)\Delta_F(x_2-x_3) \;+\; \Delta_F(x_1-x_2)\Delta_F(x_3-x_4)] \end{aligned} \tag{70}$$

whose diagrammatic interpretation appears as (71).

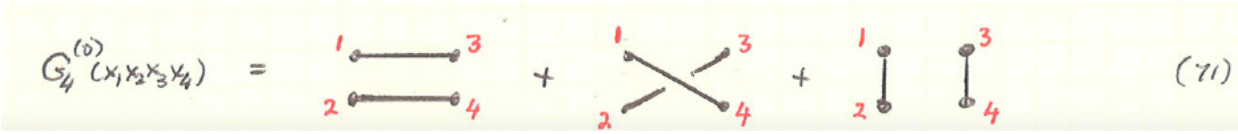
More generally we have

$$G_{2n+1}^{(0)}(x_1,\ldots,x_{2n+1}) \;=\; 0 \tag{72}$$

for correlators with odd numbers of legs, and

$$\begin{aligned} G_{2n}^{(0)}(x_1,\ldots,x_{2n}) \;&=\; \sum_p G_2^{(0)}(x_{p_1}-x_{p_2})\cdots G_2^{(0)}(x_{p_{2n-1}}-x_{p_{2n}}) \\ &=\; (i\hbar)^n \sum_p \Delta_F(x_{p_1}-x_{p_2})\cdots \Delta_F(x_{p_{2n-1}}-x_{p_{2n}}) \end{aligned} \tag{73}$$

for correlators with even numbers of legs; here $\sum_p$ means the sum over all the permutations of the indices $p_1,p_2,\ldots,p_{2n}$. This last result is actually an example of "Wick's theorem",



which in the older canonical formulation of field theory, using field operators and their commutation relations, is actually quite clumsy to prove.

So far what we have done is fairly formal. It is now time to turn to a specific theory, and take a quick glance at two of the recipes used to deal with interactions in this theory, viz., the ϕ^4 theory.

B.1.5 INTERACTIONS and $\mathcal{Z}[J]$ in ϕ^4 THEORY

We have seen that the free field theory is simple to solve because we only have to deal with Gaussian integrals. This feature disappears as soon as we introduce interactions. However, we may use the some techniques to deal with these as we did for the simple 1-particle Green function for non-relativistic particle mechanics. What we will do in the following is to derive 2 results for the full generating functional $\mathcal{Z}[J]$ in ϕ^4 theory. The derivations will be short because they parallel very closely those in ordinary QM.

RESULT 1: FUNCTIONAL DERIVATIVE $\frac{\delta}{\delta J(x)}$

Let us recall the simple result for functional integrals, discussed in the Appendix and already used in eq. (69) of the section on correlations in QM, that for a quadratic functional $Q_J[\phi]$ for the field $\phi(x)$ and a polynomial $P[\phi(x)]$ in $\phi(x)$, we have

$$\int D\phi(x)P[\phi]\, e^{-Q[\phi]} \;=\; P(-\frac{\delta}{\delta J})\int D\phi(x)\, e^{-Q_J^0[\phi]} \tag{74}$$

where $Q_J^0[\phi]$ has the general form

$$Q_J^0[\phi] \;=\; \frac{i}{\hbar}(\phi, Q_0\phi) + J\phi + C \tag{75}$$

We now assume that the polynomial in question is just the exponential functional over the interaction term in the Lagrangian. Thus, suppose we have an action

$$S[\phi; J] \;=\; \int d^Dx \left\{ -\frac{1}{2}\phi(\partial^2 + m^2)\phi - V(\phi) + J(x)\phi(x) \right\} \tag{76}$$

where in the ϕ^4 theory,

$$V(\phi) \;=\; \frac{g}{4!}\phi^4(x) \tag{77}$$

Then our polynomial is just the exponential, i.e.,

$$P[\phi] \;=\; \exp\left\{\frac{-i}{\hbar}\int d^D x\; V\left(\phi(x)\right)\right\} \;\rightarrow\; \exp\left\{\frac{-i}{\hbar}\frac{g}{4!}\int d^D x\; \phi^4(x)\right\} \tag{78}$$

so that, from eqs. (13), (14), and (74), we have the normalized $Z[J]$ in the form

$$\mathcal{Z}[J] \;=\; \frac{e^{\frac{-i}{\hbar}\int d^D x V\left(-i\hbar\frac{\delta}{\delta J(x)}\right)}\; \mathcal{Z}_0[J]}{\left[e^{\frac{-i}{\hbar}\int d^D x V\left(-i\hbar\frac{\delta}{\delta J(x)}\right)}\; \mathcal{Z}_0[J]\right]\Big|_{J=0}} \tag{79}$$

with $Z_0[J]$ given by eq. (56); thus, for the ϕ^4 theory, we have

$$Z[J] \;=\; \frac{e^{\frac{-i}{\hbar}\int d^D x \frac{g}{4!}\left(-i\hbar\frac{\delta}{\delta J(x)}\right)^4}\quad e^{\frac{-i}{2\hbar}\int d^D x_1 \int d^D x_2\; J(x_1)\Delta_F(x_1-x_2)J(x_2)}}{\left[e^{\frac{-i}{\hbar}\int d^D x \frac{g}{4!}\left(-i\hbar\frac{\delta}{\delta J(x)}\right)^4}\quad e^{\frac{-i}{2\hbar}\int d^D x_1 \int d^D x_2\; J(x_1)\Delta_F(x_1-x_2)J(x_2)}\right]\Big|_{J=0}} \tag{80}$$

which is a result that can be systematically expanded in powers of g, to give a perturbative theory for the effects of the ϕ^4 interaction. The same kind of expansion can then be given for the correlation functions, starting from the expansion for $Z[J]$. We will look at this in our section on perturbation and diagrammatic expansions, applied to the ϕ^4 theory (section B3).

RESULT 2: FUNCTIONAL DERIVATIVE $\frac{\delta}{\delta\phi(x)}$

As we already saw in dealing with ordinary QM, we can also transform expressions like eqs. (79) and (80) into expressions involving functional derivatives with respect to $\phi(x)$, as opposed to $J(x)$. This relies on the identity, discussed in the Appendix on functionals, given by

$$F\left[-i\frac{\delta}{\delta J(x)}\right]\;\; G\left[J(x)\right] \;=\; G\left[-i\frac{\delta}{\delta\phi(x)}\right]\;\; F(\phi)\; e^{i\int dx\;\phi(x)J(x)}\Big|_{\phi=0} \tag{81}$$

for 2 functionals F and G of the functions $\phi(x)$ and $J(x)$. This means we can write

$$e^{\frac{-i}{\hbar}\int d^D x V\left(-i\hbar\frac{\delta}{\delta J(x)}\right)}\; \mathcal{Z}_0[J] \;=\; \mathcal{Z}_0\left[-i\hbar\frac{\delta}{\delta\phi(x)}\right] e^{\frac{-i}{\hbar}\int d^D x[V(\phi)-\phi(x)J(x)]}\Big|_{\phi=0} \tag{82}$$

from which

$$\mathcal{Z}[J] \;=\; \frac{\mathcal{Z}_0\left[-i\hbar\frac{\delta}{\delta\phi(x)}\right]\quad e^{\frac{-i}{\hbar}\int d^D x[V(\phi)-\phi(x)J(x)]}\Big|_{\phi=0}}{\mathcal{Z}_0\left[-i\hbar\frac{\delta}{\delta\phi(x)}\right]\quad e^{\frac{-i}{\hbar}\int d^D x V(\phi(x))}\Big|_{\phi=0}} \tag{83}$$

i.e., we have converted the whole expression to one in which functional derivatives act directly on the fields $\phi(x)$ themselves, rather than on the currents. For the ϕ^4 theory with the interaction in eq. (77), eq.(83) reduces to

$$\mathcal{Z}[J] \;=\; \frac{e^{\frac{-i}{2\hbar}\int d^D x_1 \int d^D x_2\; \Delta_F(x_1-x_2)\frac{i\hbar\delta}{\delta\phi(x_1)}\frac{i\hbar\delta}{\delta\phi(x_2)}}\quad e^{\frac{-i}{\hbar}\int d^D x\left[\frac{g}{4!}\phi^4(x)-J(x)\phi\right]}\Big|_{\phi=0}}{e^{\frac{-i}{2\hbar}\int d^D x_1 \int d^D x_2\; \Delta_F(x_1-x_2)\frac{i\hbar\delta}{\delta\phi(x_1)}\frac{i\hbar\delta}{\delta\phi(x_2)}}\quad e^{\frac{-i}{\hbar}\int d^D x \frac{g}{4!}\phi^4(x)}\Big|_{\phi=0}} \tag{84}$$

with again a perturbative expansion in g in the offing; both for $\mathcal{Z}[J]$ and for the correlation functions.

B.1.6 PHYSICAL MEANING of THESE FUNCTIONS

We will only learn the full meaning of the various functions introduced here as we go along, gaining experience with them and seeing how they are deployed. However we can learn a lot by comparing what we have with other similar mathematical structures, and look at their physical meaning. We can in particular look at the analogies with thermodynamics and statistical mechanics, on the other hand, and with the ordinary problem of a single QM particle in a "noise field", on the other.

In what follows we will (i) look at the analogies with thermodynamics and statistical mechanics (and at the same time give a more complete discussion of connected graphs and $W[J]$), and then (ii) compare results here with those for 1-particle Green function. For a discussion of the relationship with noise in probability theory, go to the appendix on this topic.

(a) CORRESPONDENCE with STATISTICAL MECHANICS

At the very beginning of the section, we saw the simple way in which the structure of ordinary probability functions corresponds to the generating functional of QFT. We can make this correspondence more precise in one of two ways, i.e.,

(i) Reduce the QFT to a simple theory of functions by taking the classical limit, i.e., by restricting ourselves purely to the classical solution $\bar{\phi}(x)$ that minimizes the action.

(ii) By generalizing the simple probability theory given earlier, from a theory dealing with probabilities of outcomes $P(\phi)$ for some random variable ϕ, to probabilities for outcomes $P[\phi(x)]$ of some random process $\phi(x)$, itself a function of some variable x. The outcomes $P[\phi(x)]$ are functional of $\phi(x)$, whereas $P(\phi)$ is a function of ϕ.

The correspondence with statistical mechanics or thermodynamics comes because these subjects deal precisely with probabilities for outcomes. Ordinary thermodynamics deals with simple functions like the free energy $F(T)$, a function of temperature T; statistical mechanics, on the other hand, deals with functionals like the free energy functional $F[\phi]$ of some configuration $\phi(x) = \phi(r, t)$ of the system. In a non-relativistic condensed matter system this configuration could be written in terms of the quantum field describing the system, with probabilities $P[\phi(x)]$ for different such configurations; or it might be written in terms of the probabilities $P[\psi_j(x)]$ for the system to be in different eigenstates of the Hamiltonian - it amounts to the same thing. In a relativistic QFT we will always deal with field configurations.

In what follows we will make things simple by focussing on the classical limit. Later, when we have gotten used to the tools of QFT, we will look in a little more detail at the

SEMICLASSICAL Q.F.T.	STATISTICAL MECHANICS/THERMODYNAMICS
$Z_{cl}(J)$ Generating Function.	$Z[T]$ or $Z(H)$ Partition function
$\bar{\phi}$; J (field ; external current)	S ; T (entropy ; temperature) $\underline{M}$, $\underline{H}$ (magnetization ; external field).
$W(J) = -i\hbar \ln Z_{cl}(J)$ $= \Gamma_{cl}(\bar{\phi}) - \int d^D x\, \bar{\phi}(x) J(x)$	$F(T) = -kT \ln Z(T)$ $= E(S) - TS$ $F(H) = -kT \ln Z(H)$ $= G(M) - \int d^D r\, \underline{M}(r) \cdot \underline{H}(r)$

(88)

correspondence between QFT and full-blooded statistical mechanics.

The classical limit of a QFT for a simple scalar ϕ-field is defined by minimizing the action, i.e., for an action

$$S[\phi, J] \;=\; S[\phi] \;-\; \int d^D x J(x)\phi(x) \tag{85}$$

We define the classical solution $\bar{\phi}_J(x)$ by

$$\left.\frac{\delta S[\phi, J]}{\delta \phi(x)}\right|_{\phi=\bar{\phi}} \;=\; \left.\frac{\delta S[\phi]}{\delta \phi(x)}\right|_{\phi=\bar{\phi}} \;+\; J(x) \tag{86}$$

The generating functional $Z[J]$ then becomes a generating function, i.e.,

$$\mathcal{Z}[J] \;\to\; Z_{cl}(J) \;=\; e^{\frac{i}{\hbar}\left[S_{cl}(\bar{\phi}) + \int d^D x \bar{\phi}(x) J(x)\right]} \tag{87}$$

where we have lost the functional integration over paths $\phi(x)$ because there is now only one path $\bar{\phi}(x)$, and the action $S[\phi] \to S_{cl}(\bar{\phi})$, for that path.

Now there is a very precise correspondence between the functions $Z_{cl}(J)$, $W_{cl}(J)$, and $\Gamma_{cl}(\bar{\phi})$, on the one hand, and the thermodynamics function $F(T)$ or $F(H)$ (free energy as a function of temperature T or applied field H), the partition functions $Z(T)$ and $Z(H)$, and the functions $U(S)$ (internal energy as a function of entropy S) and $G(M)$ (thermodynamics Gibbs free energy as a function of magnetization), on the other hand. These links can be summarized in the table (88): Thus we see that in these cases, the generator $W_{cl}(J)$ of connected graphs corresponds to the free energy $F(T)$ or $F(H)$, both functions of "external" (and intensive) degrees of freedom, like J. On the other hand, a Legendre transformation to extensive degrees of freedom like S and M corresponds to the transformation of the field theory to $\Gamma_{cl}(\phi)$, a function of the field itself, which is also an extensive variable.

This analogy between statistical mechanics and QFT is one we will pursue quite often, with increasing sophistication. But notice the key difference here, which is seen in the correspondence

$$i\hbar \;\to\; kT \tag{89}$$

between the two. This difference, between real and imaginary quantities, is in one way quite crucial - probabilities in the statistics mechanical system correspond to amplitudes in the QM or QFT system. One can of course change this by rotating to imaginary time, to get a Euclidean QFT - we will also look at this later on.

We have already explored, in our discussion of 1-particle QM and correlation functions, the connection between $G(0,0|J)$, or $Z[J]$, for a particle in the presence of a "noise source" $J(t)$, and the path integral expression for this propagator. Without going into too many details (which we will do later), it is very revealing to look at the result for a harmonic oscillator subjected to a noise force $J(t)$. As we saw, this is given by the AMPLITUDE ("vacuum correlator"):

$$Z[J(t)] \;=\; e^{\frac{-i}{2\hbar}\int dt \int dt' J(t) D_0(t-t') J(t')} \tag{90}$$

where $D_0(t)$ is the simple SHO correlator. Now consider the expression which gives the PROPAGATOR for some random process $\xi(t)$ to follow some particular "path". This is described by a generating functional

$$Z[k(t)] \;=\; \int D\xi(t) P[\xi(t)]\; e^{i\int dt\; \xi(t) k(t)} \tag{91}$$

where the probability for a given $\xi(t)$ is given, for a Gaussian random walk, by

$$P[\xi] \;=\; e^{-\frac{i}{2}\int dt \int dt'\; \xi(t)\; k_0^{-1}(t-t')\; \xi(t')} \tag{92}$$

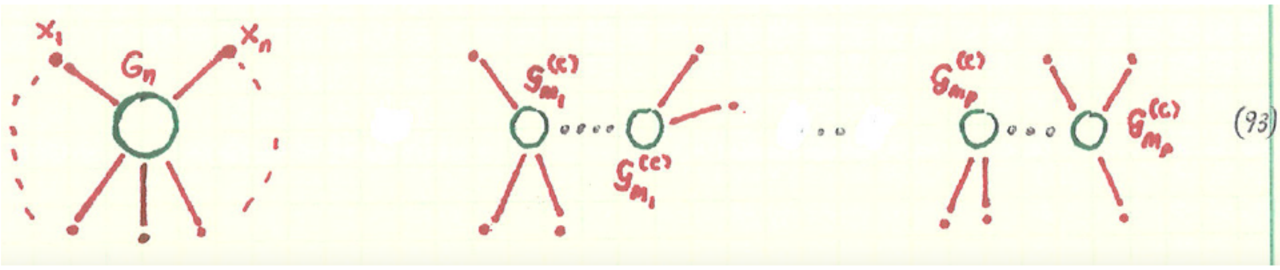
where $k_0^{-1}(t-t')$ is some correlator. We see already how we can start to make connection between random walks, constrained in one way or another, and QFT - again, we will explore this more later.

(b) CONNECTED GRAPHS

Finally, let's briefly sketch the connection between the functional $W[J]$ and the connected graphs - this also helps us understand the meaning of the functions and functionals we have defined. Consider first the relationship between the set of all connected n-point graphs, which we have called $\mathcal{G}_n^{(c)}(x_1, \ldots, x_n)$, and the set of all graphs with n external legs, which we have called $G_n(x_1, \ldots, x_n)$. This relationship is intuitively rather obvious, and we have already seen it shown graphically in (37). Clearly we can get all graphs for $G_n(x_1, \ldots, x_n)$ by taking the set of all graphs for $\mathcal{G}_{m_1}^{(c)}(x_1, \ldots, x_{m_1})$, $\mathcal{G}_{m_2}^{(c)}(x_1, \ldots, x_{m_2})$, $\ldots$, $\mathcal{G}_{m_p}^{(c)}(x_1, \ldots, x_{m_p})$, such that $\sum_{j=1}^{p} m_j = n$, and combining them in all possible ways, with all possible choices for $m_1, \ldots, m_p$ summed over.

For those who do not trust the graphical argument given in (37), a direct calculation may be more convincing. So let's do this. Schematically we have the graphs in (93) where we imagine graphs for $\mathcal{G}_{m_j}^{(c)}$ repeated q_j times, and so on, and fix the condition that

$$\sum_{j=1}^{p} q_j m_j \;=\; n \tag{94}$$



Now we just have a combinational problem. At order n, the number of different graphs is

$$N_n \;=\; \frac{n!}{(m_1!)^{q_1} q_1! \cdots (m_p!)^{q_p} q_p!} \tag{95}$$

and then summing over all n, we get for $\mathcal{Z}[J]$ the expression

$$\begin{aligned}\mathcal{Z}[J] \;=\; & \sum_n \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int d^D x_1 \cdots \int d^D x_n J(x_1) \cdots J(x_n) \\ & \times \sum_{q_1 m_1 + \ldots q_p m_p = n} \mathcal{G}^{(c)}_{m_1}(x_1, \ldots, x_{m_1}) \cdots \mathcal{G}^{(c)}_{m_p}(x_{1_p}, \ldots, x_{m_p})\end{aligned} \tag{96}$$

which we write as

$$\begin{aligned}\mathcal{Z}[J] \;&=\; \sum_n \left(\frac{i}{\hbar}\right)^n \sum_{q_1 m_1 + \ldots q_p m_p = n} \prod_{j=1}^{p} \frac{1}{q_j!} \left[\frac{\int d^D x_1 \cdots \int d^D x_{m_j}\, \mathcal{G}^{(c)}_{m_j}(x_1, \ldots, x_{m_j})\; J(x_1) \cdots J(x_{m_j})}{m_J!}\right]^{q_j} \\ &=\; \sum_{q_j} \prod_j \left(\frac{i}{\hbar}\right)^{q_j} \frac{1}{q_j!} \left[\frac{\int d^D x_1 \cdots \int d^D x_{m_j}\, \mathcal{G}^{(c)}_{m_j}(x_1, \ldots, x_{m_j})\; J(x_1) \cdots J(x_{m_j})}{m_J!}\right]^{q_j} \\ &=\; \exp\left[\sum_{n=1}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int d^D x_1 \cdots \int d^D x_n\, \mathcal{G}^{(c)}_n(x_1, \ldots, x_n)\; J(x_1) \cdots J(x_n)\right]\end{aligned} \tag{97}$$

so that, comparing with (31), we get again that

$$\mathcal{Z}[J] \;=\; e^{\frac{i}{\hbar} W[J]} \tag{98}$$

which demonstrates again the relationship we were after.

We see that by going through exactly the same argument, but now functionally differentiating both sides n times with respect to $J(x_j)$, and then taking $J \to 0$, we get the same result as that quoted in eqtn (37) for the relation between the full correlators $G_n(x_1 \cdots, x_n)$ and the connected correlators $\mathcal{G}^{(c)}_n(x_1, \cdots, x_n)$.