Phys 508: HOMEWORK ASSIGNMENT No (3)

Saturday March 25th 2017

DUE DATE: Monday April 17th 2017.

Assignments handed in late will not receive a full mark.

QUESTION (1) Electron Self-energy in QED: Functional Approach: The computation of the lowest-order self-energy graph in QED is a well-known exercise. Here we do the full non-perturbative calculation of $\Sigma(p)$ at low momentum (ie., $p^2 \ll m^2$) using the functional WKB method of the notes. This means writing the fermion propagator as

$$\mathcal{G}_2\left(x, x'\right) \;=\; \int \mathcal{D}A^\mu \; e^{\frac{i}{2\hbar}\sum_q \left[A^\mu_q \left(D^0_{\mu\nu}(q)\right)^{-1} A^\nu_{-q}\right]} \; \mathcal{G}_2\left(x, x'|A\right) \tag{1}$$

where $\mathcal{G}_2\left(x, x'|A\right)$ is the fermion propagator in the presence of a "frozen" field $A^\mu(x)$; and then carrying out the functonal integral.

(a) First we need to find an expression for $\mathcal{G}_2\left(x, x'|A\right)$, which satisfies the equation:

$$\left[\gamma^\mu\left(i\partial_\mu + eA_\mu\right) - m\right] \; \mathcal{G}_2\left(x, x'|A_\mu\right) \;=\; -\delta\left(x - x'\right) \tag{2}$$

By doing a partial Fourier transform over the variable $x - x'$, and then using the proper time representation $\mathcal{G}_k\left(x; s|A\right) \;=\; \exp\{i\frac{s}{\hbar}\left(\gamma^\mu p_\mu - m + i\delta\right)\}e^{i\Psi_s(x|A)}$, derive the differential equation for the exponential factor Ψ_s. Then make a WKB expansion for Ψ_s, as a sum over terms $\psi_n(s)$, and find the set of WKB-style recursion equations linking the different $\{\phi_n\}$.

(b) Using the recursion equation derived above, show that the lowest term $\psi_1(s; x|A)$ in the series expansion for Ψ_s has the solution

$$\psi_1\left(s; x|A\right) = \int_0^s ds' \sum_q e^{-iqx} e^{is'p^\mu q_\mu} \; e p_\mu A^\mu\left(q\right) \tag{3}$$

where $A_\mu(q)$ is the Fourier transform of $A_\mu\left(x\right)$.

(c) Now, do the functional integration over the gauge field $A^\mu(q)$, using your answer for $\psi_1\left(s; x|A\right)$ derived above, to find the result

$$\mathcal{G}_2\left(p\right) \;=\; i\int_0^\infty ds \, \exp\left\{ i\left[\gamma^\mu p_\mu - m + i\delta\right] - e^2 \sum_q \frac{1 - e^{isp^\mu q_\mu}}{\left(p^\mu q_\mu\right)^2} p^\mu D^0_{\mu\nu}\left(q\right) p^\nu \right\} \tag{4}$$

for the 1-fermion propagator.

Evaluation of the 4-d integrals then allows us to find $\Sigma(p)$.

QUESTION (2) Variations on Kosterlitz-Thouless: Here we study what happens to the usual XY/KT model when we slightly change things - this gives a better appreciation of its structure.

(a) Let us consider the problem of a KT system in dimension $d = 2 + \epsilon$. The first thing to do is rederive the scaling laws for this case. Show that

$$\begin{aligned} \frac{dT}{d\ell} &= -\epsilon T + 4\pi^3 y^2 \\ \frac{dy}{d\ell} &= (2 - \pi/T)y \end{aligned} \tag{5}$$

where T is the temperature, y is the fugacity, and ℓ the usual length parameter.

(b) We now wish to use these scaling laws, in a way analogous to that in $d = 2$. From the scaling laws just derived:

(i) find the new position of the fixed point for the KT transition (compared to $d = 2$).

(ii) Find the eigenvalues at the fixed point to lowest order non-trivial order in ϵ.