# Phys 508: HOMEWORK ASSIGNMENT No (2)

Monday March 6th 2017

**DUE DATE: Monday March 20th 2017.**

Assignments handed in late will not receive a full mark.

This assignment is mostly to do with equations of motion for various non-linear field theories (along with different kinds of expansion); we pick two examples.

## QUESTION (1): $\phi^6$ THEORY

Consider the theory of a massive relativistic scalar field containing an interaction which is 6-th order in the field, so that the action is

$$S = \int d^4x \left[ \frac{1}{2}(\partial^\mu \phi \partial^\mu \phi - m^2\phi^2) - \frac{\lambda_6}{6!}\phi^6 \right] \tag{1}$$

**(i)** Draw all graphs up to 2nd order in $\lambda_6$ for the connected correlators $\mathcal{G}_2, \mathcal{G}_4$, and $\mathcal{G}_6$. Then, consider the 2nd-order graph for the 4-point vertex $\Gamma_4(1,2,3,4)$ which contains 4 internal lines connecting the interaction vertices. Write the explicit expression for this graph, in both real spacetime and in 4-momentum space. Finally, make a table showing all graphs up to $\sim O(\lambda_6^2)$ for the 6-point, 8-point, and 10-point vertices in the theory; give the expressions for each one as integrals over Feynman propagators, and specify the order in $\hbar$ and the number of loops $L$.

**(ii)** Derive the hierarchy of Schwinger-Dyson equations for this theory. You should find the general expression, showing how you get it, and also list the equations in this hierarchy up to all equations that include $\mathcal{G}_4$. Then, show what form the Bethe-Salpeter equation will take for this theory, for the 4-point vertex $\Gamma_4$, in terms of irreducible vertices. You should show your result graphically, and give the expression for it both real spacetime and in 4-momentum space.

**(iii)** Find the equation of motion for the classical solution $\Phi_o(x)$ of this field theory (ie., the solution when $\hbar = 0$). Then, writing the full solution as $\phi(x) = \Phi_o(x) + \psi(x)$, find the action in terms of $\Phi_o(x)$ and this "quantum fluctuation field" $\psi(x)$, and derive an expression for both the classical action and the shift in this action caused by the quantum fluctuations. Show the graphs for the various vertices in this theory, and by integrating over the fluctuations, evaluate the lowest correction to the classical action from these fluctuations.

**(iv)** Now let us consider the zero-dimensional limit of this $\phi^6$ theory. Consider an 'amplitude' function $A(\phi) = e^{iS(\phi)/\hbar}$ of a simple variable $\phi$, where the exponent $S(\phi) = -(m^2\phi^2/2 - \frac{\lambda_6}{6!}\phi^6)$. We define a generating functions $Z(J) = \int d\phi A(\phi) e^{i\phi J}$ and 'correlators'

$g_n$ by the power series expansion

$$Z(J) = \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!}\, g_n J^n \tag{2}$$

Find the form of $Z_o(J)$, the 'free distribution' with $\lambda_6 = 0$; and show that for $Z_o(J)$, the free correlators $g_{2n}^{(o)} \propto (2n-1)!!/m^{2n}$, where $(2n-1)!! = (2n-1)(2n-3).. \times 3 \times 1$. Then derive a differential equation for $Z(J)$ in terms of differential operators $d/dJ$ acting on $Z_o(J)$.

## QUESTION (2): Sine-Gordon model

The standard Sine-Gordon model has the action

$$S = \int dt dx \left( \frac{1}{2} \left[ (\partial_t \theta)^2 - c_o^2 (\partial_x \theta)^2 \right] \; + \; \omega_o^2 (\cos\theta - 1) \right) \tag{3}$$

for the field $\theta(x,t)$ in 1+1 dimensions.

**(i)** Derive the 1-soliton solutions $\psi_{\pm}(x,t)$ for this model.

**(ii)** Now write the field $\theta(x,t)$ around one of these solutions as $\theta_+(x,t) = \psi_+(x) + \varphi(x,t)$, where $\varphi(x,t)$ is a small deviation around the soliton, and where for simplicity we choose a static soliton solution at the origin. From this derive an equation of motion for $\varphi(x,t)$ up to <u>2nd order</u> in $\varphi(x,t)$ (ie., one order beyond the simple linearized theory). Then write the effective action for the $\varphi$-field which corresponds to this equation of motion; and show the bare vertices that exist in this theory, along with the values to be attached to each of them.

**(iii)** From the results just derived you can derive a self-energy correction for the soliton, caused by its interaction with fluctuations. Show the lowest-order contributions to this self-energy as graphs, and then write down the expressions for these graphs (there is no need to evaluate them however!).