Phys 508: HOMEWORK ASSIGNMENT No (1)

Sunday Feb 5th 2017

DUE DATE: Monday Feb 27th 2017.

Assignments handed in late will not receive a full mark.

This assignment is mostly to do with functional methods and with graphical calculations, both with relativistic scalar fields and with non-relativistic fermions and bosons.

QUESTION (1): COUPLED SCALAR FIELDS

Consider a pair of relativistic coupled massive scalar fields with the action

$$S[\phi, \chi] \;=\; \int d^4x \, \frac{1}{2} \left\{ \left[\partial_\mu \phi \partial^\mu \phi - m_0^2 \phi^2\right] + \left[\partial_\mu \chi \partial^\mu \chi - M^2 \chi^2 - \frac{2g}{4!} \chi^4 \right] - \lambda_0 \phi^2(x) \chi(x) \right\}$$

with interfield coupling vertex $-i\lambda_0/2\hbar$, and a self-coupling vertex $-(ig/\hbar)/4!$ for the χ-field.

(i) By adding external currents $J(x), \mathcal{I}(x)$ to the generating functional for this theory, coupling to $\phi(x)$ and $\chi(x)$ respectively, find the two Schwinger-Dyson equations for the theory, one for each field, giving explicit expressions for the relevant functional derivatives of the action.

(ii) Give the diagram rules for the free field propagators and the vertices for this theory. Then show all diagrams for (a) the ϕ-field self-energy, and (b) the χ-field self-energy, up to 4th-order in the combined couplings (ie., including diagrams up to order g^4, λ_o^4, and $\lambda^2 g^2$).

(iii) Now we are going to derive an effective action for the system. We first "freeze" the χ-field, writing $\chi(x) \to \chi_0(x)$, where $\chi_0(x)$ is some fixed configuration. Now show that we can write the generating functional in the form

$$\mathcal{Z}[J, \mathcal{I}] \;=\; \int \mathcal{D}\chi \int \mathcal{D}\chi_o \; \delta(\chi - \chi_0) \; e^{\frac{i}{\hbar}(S_\chi[\chi_0] - \int \mathcal{I} \chi_0)} \; \tilde{\mathcal{Z}}[\chi_o, J]$$

where S_χ is the action for the χ-field, and $\tilde{\mathcal{Z}}[\chi_o, J]$ is the result of integrating over the ϕ-field at fixed χ_0; you should derive the form of $\tilde{\mathcal{Z}}[\chi_o, J]$.

(iv) Finally, show how we can write $\tilde{\mathcal{Z}}[\chi_o, J]$ in a diagrammatic expansion in powers of the vertex $V_0 = -i\lambda_o \chi_0 / 2\hbar$; and show the diagrams contributing to this expansion up to order V_0^2.

QUESTION (2): MATSUBARA DIAGRAMS

We look at the simple non-relativistic Fermion Loop, or "particle-hole bubble", at finite T, in the Matsubara formalism; we will assume spin-1/2 free fermions in a system that is isotropic and translationally invariant. We will also apply a magnetic field, so that the particle dispersion is just $\epsilon^o_{\mathbf{k}\sigma} = \hbar^2|\mathbf{k}|^2/2m - \sigma\gamma H$, where $\sigma = \pm$, and γ is a constant; and we write the Fermi distribution function as $f_{\mathbf{k}\sigma} = f(\epsilon^o_{\mathbf{k}\sigma})$.

(i) Recall that for continuum spinless fermions, the pair bubble is given by

$$\pi_o(\mathbf{q}, z) \;=\; \sum_k \frac{f_k - f_{k+q}}{z - (\epsilon_{k+q} - \epsilon_k)}$$

For a spin-1/2 fermion system in a finite field H, find the analogous expressions for $\pi_o^{\sigma\sigma'}(\mathbf{q}, z) = \int d\epsilon \sum_k G_o^\sigma(k+q, \epsilon + z) G_o^{\sigma'}(k, \epsilon)$ for the cases $\sigma = \sigma'$, and $\sigma = -\sigma'$. Neglect any orbital effects from the magnetic field - you can assume the fermions are electrically neutral.

(ii) Now do the same for the case of a finite 1-d system in a field H, with fermions confined between 2 plates a distance L apart, and boundary conditions $\psi(x=0) = \psi(z=L) = 0$, to find an expression for the vacuum polarization function $\pi_o^{\uparrow\uparrow}(z, q_l)$, where q_l refers to the wave-vector associated with the l-th confined state. Show that it can be brought to the form of a sum proportional to $\sum_n (F(z,l,L) - n^2)^{-1}$, where $F(z,l,L)$ is a specific function of its arguments.

(iii) Now consider the case where $L \to \infty$ (ie., the 1-dimensional continuum limit), and do the momentum integral to find an explicit expression for $\pi_o^{\uparrow\uparrow}(q, z)$

QUESTION (3): VERTEX CORRECTIONS

We consider the 2 graphs shown in the figures (Figs 1 and 2) - they are the lowest order vertex corrections for (i) a Fermi system with instantaneous 4-point interactions V_o, and (ii) a set of fermions interacting via bosonic phonon-like excitations. We wish to evaluate these graphs by doing the frequency sums - we will not evaluate the momentum sums.

(i) Calculate and expression for the first graph (Fig 1) at finite T, involving instantaneous interactions V_o, using the Matsubara formalism.

You can assume that the external boson line is a phonon, coupled to the fermion via a coupling constant λ_o.

(ii) Consider the 2nd graph shown in Fig. 2. Calculate this graph at finite T, doing only the frequency sums, and assuming the Hamiltonian

$$H \;=\; \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \;+\; \sum_{\mathbf{q}} \hbar\Omega_{\mathbf{q}} (b^\dagger_{\mathbf{q}} b_{\mathbf{q}} + \frac{1}{2}) \;+\; \lambda_o \int d^3\mathbf{r} \psi^\dagger_\sigma(\mathbf{r}) \psi_\sigma(\mathbf{r}) \phi(\mathbf{r}) \tag{5}$$

with coupling λ_o. Again, as in part (i), the external bosonic line is a phonon, coupled to the fermion via a coupling constant λ_o.

QUESTION (4): GAUGE THEORIES

In ordinary QED, the field strength is defined in terms of the vector potential by $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$.

(i) Show that $F_{\mu\nu}(x)$ is invariant under the gauge transformation $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \theta(x)$, and that

$$\partial_\alpha F_{\mu\nu}(x) + \partial_\nu F_{\alpha\mu}(x) + \partial_\mu F_{\nu\alpha}(x) \; = \; 0$$

which is just the 'Bianchi identity' for electrodynamics.

Then show that the Dirac matter Lagrangian, viz., $L = \bar{\psi}(x)[i\gamma^\mu \partial_\mu - m]\psi(x)$, is not invariant under the transformation $\psi(x) \rightarrow e^{ie\theta(x)}\psi(x)$, where e is the charge, but that a simultaneous gauge transformation on $\psi(x)$ and $A_\mu(x)$ leads to a gauge-invariant form for the combined action of matter and gauge fields.

(ii) Now consider a theory with massive photons, so that the free photons have action

$$S[A] \; = \; \int d^4x \; A_\mu(x) \; [(\partial^2 + m^2)\eta^{\mu\nu} - \partial^\mu \partial^\nu] \; A_\nu(x) \qquad (1)$$

Show that this action is *not* gauge invariant. Then derive the expression for the free generating functional $\mathcal{Z}_o[J^\mu]$ defined by coupling $A_\mu(x)$ to an external source $J^\mu(x)$, and give the form for the propagator $D^o_{\mu\nu}(q)$ for the photon. By Fourier transforming this in the static limit (where the frequency of the photon $\rightarrow 0$), find the new form of the Coulomb interaction between particles coupling to $A_\mu(x)$ via an interaction $\int eA_\mu(x)J^\mu(x)$.

(iii) Now let us consider the equation of motion for a spin-2 bosonic field $h_{\mu\nu}(x)$, a tensor field, which can be shown to take the form

$$\partial^2 h_{\mu\nu}(x) - 2\partial^\alpha \partial_\nu \bar{h}_{\mu\alpha}(x) = -2\lambda \bar{T}_{\mu\nu}(x)$$

where $T_{\mu\nu}(x)$ is the energy-momentum tensor, and the bar symbol over a tensor quantity $t_{\mu\nu}$ denotes $\bar{t}_{\mu\nu} = t_{\mu\nu} - \frac{1}{2}t\eta_{\mu\nu}$, where $t = t^\alpha_\alpha$, ie., we use the summation convention.

Show that if we make the gauge transformation $h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + [\partial_\mu \chi_\nu(x) + (\partial_\nu \chi_\mu(x)]$, then the equation of motion is unaffected. Then show that if we make the gauge choice that $\partial_\alpha \bar{h}^{\mu\alpha} = 0$, we get the equation of motion $\partial^2 h_{\mu\nu}(x) = -2\lambda \bar{T}_{\mu\nu}(x)$.

See Figures on next page

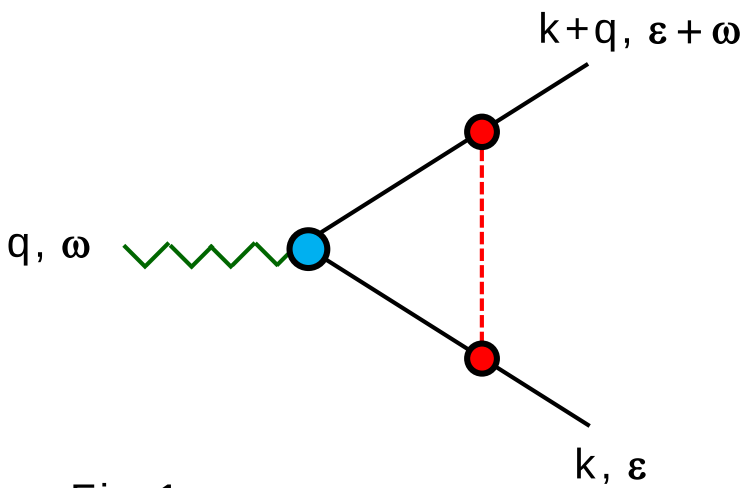

Fig 1

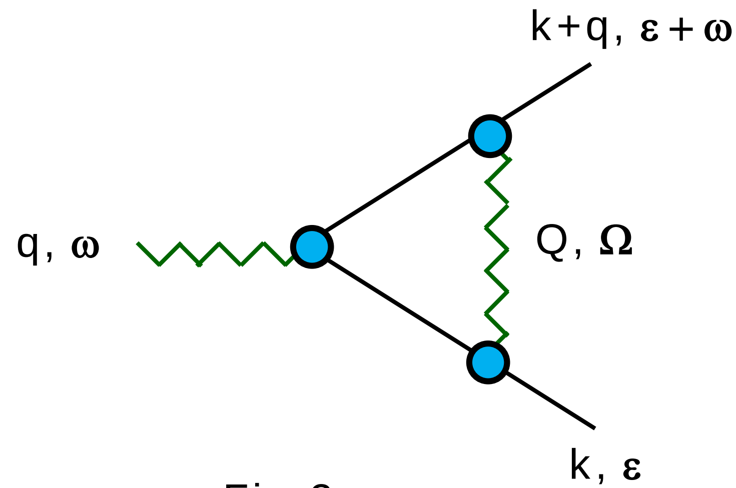

Fig 2