Phys 508: HOMEWORK ASSIGNMENT No (5)

Friday 25th March, 2016

DUE DATE: Friday April 15th 2016

Assignments handed in late will not receive a full mark.

QUESTION (1): GRADIENT EXPANSIONS

1.A: GRADIENT EXPANSION OF PROPAGATOR We will consider a simple field theory with a real field $\phi(x)$ and with action

$$S\left[J\right]=\int d^4x\,\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi-m^2\phi^2\right)-V\left(x\right)\phi^2\left(x\right) \tag{1}$$

so that the propagator $\mathcal{G}\left(x,x'|J\right)$ defined in the presence of the field $V(x)$ satisfies the equation $\left(\partial^2+m^2+V\left(x\right)\right)\mathcal{G}\left(x,x'|J\right)=-\delta\left(x-x'\right)$. A partial Fourier transform of this equation, assuming $\mathcal{G}\left(x,x'|V\right)$ varies slowly with respect to the "centre of mass" coordinate $(x+x')/2$, but quickly with respect to the relative coordinate $x-x'$, is written in the proper time WKB representation as

$$\mathcal{G}_k\left(x|J\right)=i\int_0^\infty ds\,e^{-is\left(k^2+m^2-i\delta\right)}\ \exp\sum_n\hbar^n\psi_n\left(k,x;s|V\right) \tag{2}$$

(i) Show that the $\psi_n\left(k,x;s|J\right)$ satisfy the hierarchy of differential equations given by

$$\left(\partial^2+2ik_\mu\partial^\mu\right)\psi_n+\sum_{r=1}^{n-1}\partial_\mu\psi_r\partial^\mu\psi_{n-r}+V\left(x\right)\delta_{n1}=-i\partial_s\psi_n \tag{3}$$

(ii) Then, by first finding $\psi_1\left(k,x;s|V\right)$, show we can write $\psi_n\left(k,x;s|V\right)$ in the form

$$\psi_n\left(k,x;s|V\right)=i\int_0^s ds' e^{i(s-s')\left(\partial^2+2ik_\mu\partial^\mu\right)}\sum_{r=1}^{n-1}\partial_\mu\psi_r\left(k,x;s'|V\right)\partial^\mu\psi_{n-r}\left(k,x;s'|V\right) \tag{4}$$

1.B: GRADIENT EXPANSION of $\mathcal{Z}_o[J]$ We can write the free field propagator $\mathcal{G}_o\left(x,x'|J\right)$ for a scalar field, in the presence of a current $J(x)$, in the form

$$\mathcal{G}_o\left(x,x'|J\right)=-\hbar^2\frac{\delta^2 Z_0[J]}{\delta J\left(x\right)\delta J\left(x'\right)} \tag{5}$$

where $\mathcal{Z}_0[J]$ is the usual bare generating functional (and we do NOT put $J=0$).

(i) Set up a gradient expansion starting from this form, either by first taking gradients of $\mathcal{Z}_o[J]$, and then functionally differentiating to get $\mathcal{G}(x, x'|J)$, or by doing these 2 operations in the opposite order.

(ii) Now try to get a gradient expansion in WKB form, starting again from $\mathcal{Z}_0[J]$.

NB: I am leaving question 1.B somewhat open-ended, so you can play around a little with these expansions.

QUESTION (2): VACUUM PERTURBATIONS and ANOMALIES

2.A: CASIMIR EFFECT for a BOX Consider a QED vacuum containing a rectangular box having perfectly conducting sides of lengths L_x, L_z, and with one of the walls parallel to the x-axis able to move without friction along the z-axis.

(i) Find the Casimir energy as a function of $L_x, L_z,$, assuming the walls are fixed.

(ii) Suppose that at $t = 0$ we allow the moveable wall to start moving from rest, with the initial $L_z = L_z^0$ assumed to be very large. Find the equation of motion, and its solution, for the coordinate $L_z(t)$ of the moveable wall, for times such that $L_z(t) > 0$ (NB: here we ignore the possibility of vacuum excitation by the moving wall).

2.B: PROPER TIME EXPANSION for the QED ANOMALY In the discussion of vacuum polarization and Schwinger vacuum instabilities, and of anomalies in chiral QED, certain identities are required; the purpose of the following is to make sure of these.

(i) Defining $D_\mu = \partial_\mu + iA_\mu$, show that $\hat{K} \equiv (\gamma^\mu D_\mu)^2 = D^2 + S^{\mu\nu} F_{\mu\nu}/2$, where $F_{\mu\nu}$ is the usual EM field tensor, and $2S_{\mu\nu} = [\gamma^\mu, \gamma^\nu]$ is the spin factor. Then show that, when $F_{\mu\nu}$ describes a static electric field of strength E_o,

$$tr[\exp i(sS^{\mu\nu} F_{\mu\nu}/2)] \;=\; 4 \cosh sE_o \tag{6}$$

Finally, show that $\int_0^\infty \frac{ds}{s}[e^{isa} - e^{isb}] = ln|b/a|$.

(ii) The proper time expansion writes the "heat kernel" propagator in the form $G_K(x, x'; s) = G_o(x, x'; s) \sum_n a_n(x, x')s^n$, where $G_K(x, x'; \tau)$ can be written as

$$G_K(x, x'; s) \;=\; \sum_p e^{ipx} e^{-\hat{K}s} e^{-ipx'} \tag{7}$$

with $\hat{K} = (\gamma^\mu D_\mu)^2$ in QED, and $G_o(x, x'; s) = \sum_p e^{ip(x-x')} e^{p^2 s}$. Find the first 3 proper time coefficients $a_n(x, x; s)$, with $x = x'$ (ie., the diagonal coefficients), for $n = 0, 1, 2$, which appear in the calculation of the QED vacuum anomaly.

END of QUESTION SHEET 5