Phys 508: HOMEWORK ASSIGNMENT No (4)

Tuesday 8th March, 2016

DUE DATE: Monday Mar 21st 2016

Assignments handed in late will not receive a full mark.

QUESTION (1): FLUCTUATION EXPANSIONS for SCALAR FIELDS

Here we look at the fluctuation expansions about the semiclassical limit for various kinds of scalar field theory.

1.A: SINE-GORDON MODEL We start in $1+1$ dimensions. The standard Sine-Gordon model has the action

$$S = \int dtdx \left(\frac{1}{2} \left[(\partial_t \theta)^2 - c_o^2 (\partial_x \theta)^2 \right] \; + \; \omega_o^2 (\cos\theta - 1) \right) \tag{1}$$

for the field $\theta(x,t)$ in 1+1 dimensions.

(i) Derive the 1-soliton solutions $\psi_\pm(x,t)$ for this model.

(ii) Now write the field $\theta(x,t)$ around one of these solutions as $\theta_+(x,t) = \psi_+(x) + \varphi(x,t)$, where $\varphi(x,t)$ is a small deviation around the soliton, and where for simplicity we choose a static soliton solution at the origin. From this derive an equation of motion for $\varphi(x,t)$ up to <u>2nd order</u> in $\varphi(x,t)$ (ie., one order beyond the simple linearized theory). Then write the effective action for the φ-field which corresponds to this equation of motion; and show the bare vertices that exist in this theory, along with the values to be attached to each of them.

1.B: LOOPS and ϕ^6 THEORY Now let us go to $3+1$ dimensions. Consider the theory of a real massive relativistic scalar field $\phi(x)$ containing interactions of 6-th order only in the field, so that the action is

$$S \;=\; \int d^4x \; \left[\frac{1}{2} (\partial^\mu \phi \partial^\mu \phi - m^2 \phi^2) \;-\; \frac{g_6}{6!} \phi^6 \right] \tag{2}$$

and we have a "ϕ^6-theory". We are going to look at the loop expansion for this theory.

(i) Make a table, analogous to that in the notes, showing all graphs up to $\sim O(g^2)$ for the 6-point, 8-point, and 10-point vertices in the theory; give the expressions for each one as integrals over Feynman propagators, and specify the order in $\hbar$ and the number of loops L.

(ii) Now find the equation of motion for the classical solution $\Phi_o(x)$ of this field theory (ie., the solution when $\hbar = 0$). Then, writing the full solution as $\phi(x) = \Phi_o(x) + \psi(x)$, find the action in terms of $\Phi_o(x)$ and this "quantum fluctuation field" $\psi(x)$, and derive an expression for both the classical action and the shift in this action caused by the quantum fluctuations. Show the graphs for the various vertices in this theory, and by integrating over the fluctuations, evaluate the lowest correction to the classical action from these fluctuations.

QUESTION (2): GAUGE INTERACTIONS

Here we shall look at the way in which gauge invariance influences interactions in coupled fermion-gauge field theories - we shall look at both spin-1 and spin-2 fields, in $3+1$ dimensions.

2.A: SPIN-1 GAUGE FIELDS In ordinary QED, the field strength is defined in terms of the vector potential by $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$; and $F_{\mu\nu}(x)$ is invariant under the gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu \theta(x)$. Now consider a theory with massive photons, so that the free photons have action

$$S[A] \;=\; \frac{1}{2} \int d^4x \; A_\mu(x) \; [(\partial^2 + m^2)\eta^{\mu\nu} - \partial^\mu \partial^\nu] \; A_\nu(x) \tag{3}$$

with photon mass m.

(i) Show that this action is *not* gauge invariant. Then derive the expression for the free generating functional $\mathcal{Z}_o[J^\mu]$ defined by a coupling $A_\mu(x)$ to an external source $J^\mu(x)$, via an interaction $\int A_\mu(x) J^\mu(x)$, and give the form for the free propagator $D^o_{\mu\nu}(q)$ for the photon.

(ii) By Fourier transforming $D^o_{\mu\nu}(q)$ in the static limit (where the photon frequency $q^o \to 0$), find the new form of the Coulomb interaction between particles coupling to $A_\mu(x)$ via an interaction via a gauge interaction $\int eA_\mu(x) j^\mu(x)$, where $j^\mu(x)$ represents an electronic fermion current.

2.B: SPIN-2 GAUGE FIELDS Now let us consider the equation of motion for a spin-2 bosonic field $h_{\mu\nu}(x)$, a symmetric tensor field, which can be shown to take the form

$$\partial^2 h_{\mu\nu}(x) - \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha}(x) - \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha}(x) = -2\lambda \bar{T}_{\mu\nu}(x)$$

where $T_{\mu\nu}(x)$ is the energy-momentum tensor, and the bar symbol over a tensor quantity $t_{\mu\nu}$ denotes $\bar{t}_{\mu\nu} = t_{\mu\nu} - \frac{1}{2} t \eta_{\mu\nu}$, where $t = t^\alpha_\alpha$, ie., we use the summation convention. This field describes gravitons, or bosonic excitations in certain solids.

(i) Show that if we make the gauge transformation $h_{\mu\nu}(x) \to h_{\mu\nu}(x) \; + \; [\partial_\mu \chi_\nu(x) + (\partial_\nu \chi_\mu(x)]$, then the equation of motion is unaffected. Then show that if we make the gauge choice that $\partial_\alpha \bar{h}^{\mu\alpha} = 0$, we get the equation of motion $\partial^2 h_{\mu\nu}(x) = -2\lambda \bar{T}_{\mu\nu}(x)$.

(ii) By Fourier transforming this equation of motion, show that there will exist an interaction between energy momentum tensor fields of form

$$V(k, k') \;=\; \frac{2T_{\mu\nu}(k) T^{\mu\nu}(k') - T(k)T(k')}{2|k-k'|^2}$$

in momentum space; and show that we can also write this interaction in the form $V(k,k') = T_{\mu\nu}(k)\mathcal{D}_o^{\mu\nu\alpha\beta}(k-k')T_{\alpha\beta}(k')$, where the spin-2 field propagator is

$$\mathcal{D}_o^{\mu\nu\alpha\beta}(q) \;=\; \frac{1}{2q^2}[\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}]$$

Now show that in the static limit, where we let the frequency component of q go to zero, the real space form of this interaction can be written as

$$V(\mathbf{r}) \;=\; -2\lambda \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{|\mathbf{q}^2 + i\delta}$$

and evaluate this 3-d Fourier transform to get the result $V(\mathbf{r}) \;=\; -\lambda/2\pi|\mathbf{r}|$.

END of QUESTION SHEET 4