Phys 508: HOMEWORK ASSIGNMENT No (1)

Friday January 19th 2016

DUE DATE: Monday Feb 1st 2015.

Assignments handed in late will not receive a full mark.

This assignment is mostly to do with revising and extending bits and pieces that were partially covered in the QFT 1 course, primarily to do with graphical calculations, and with scalar fields. Some of the questions are quite short, others longer - the very last question is the longest.

QUESTION (1): SCALAR FIELDS

1(A) Interacting Scalar Field: We are going to look at ϕ^3 theory, and start off in the zero-dimensional case. Thus we begin by considering an 'amplitude' function $A(\phi) = e^{iS(\phi)/\hbar}$ of a variable ϕ, where the exponent $S(\phi) = -(m^2\phi^2/2 + \lambda_o\phi^3/6)$. We define a generating functions $Z(J) = \int d\phi A(\phi) e^{i\phi J/\hbar}$ and 'correlators' g_n by the power series expansion

$$Z(J) = \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!}\, g_n J^n$$

1.A.(i) Find, for this zero-dimensional theory, the form of $Z_o(J)$, the 'free distribution', ie., where $\lambda_o = 0$; and show that for $Z_o(J)$, the free correlators $g_{2n}^{(o)} \propto (2n-1)!!/m^{2n}$, where $(2n-1)!! = [(2n-1)(2n-3).. \times 3 \times 1]$ is the double factorial.

Then, for finite λ_o, derive an expression for $Z(J)$ in terms of differential operators d/dJ acting on $Z_o(J)$.

1.A.(ii) Show that $Z(J)$ for this theory obeys the differential 'equation of motion'

$$\left[\frac{\lambda_o\hbar}{2}\frac{d^2}{dJ^2} \;+\; im^2\frac{d}{dJ} \;+\; \frac{J}{\hbar}\right] Z(J) = 0$$

1.A.(iii) Now we go over the full $4-d$ field theory, of a scalar field theory with action

$$S[\phi] \;=\; \int d^4x \left[\frac{1}{2}[\partial_\mu\phi(x)\partial^\mu\phi(x) - m^2\phi^2(x)] - \frac{\lambda_o}{6}\phi^3(x)\right]$$

in 4 spacetime dimensions.

Without going through the derivation, give the diagrammatic rules for the vertices and free propagator for this theory; and then draw all graphs up to 2-loop for the 2-point vertex Γ_2 and the 3-point vertex Γ_3.

1(B) Complex Scalar Field: We consider a complex field $\Phi(x)$ with components $\phi^\dagger(x)$ and $\phi(x)$, in 4d-spacetime, and having the action and generating functional

$$S[\Phi] \;=\; \int d^4x \, \frac{1}{2}(\partial_\mu \phi^\dagger \partial^\mu \phi - m^2|\phi|^2) \;-\; V(|\phi|^2)$$

$$\mathcal{Z}[J, J^\dagger] \;=\; \int \mathcal{D}\phi \int \mathcal{D}\phi^\dagger e^{\frac{i}{\hbar}\left(S[\phi,\phi^\dagger] \,+\, \frac{1}{2}\int d^4x (J\phi^\dagger + J^\dagger \phi)\right)}$$

where we have introduced two external currents $J(x), J^\dagger(x)$.

1.B.(i) Show that the generating functional for the free theory (with $V(\phi, \phi^\dagger) = 0$) is

$$\mathcal{Z}_o[J, J^\dagger] \;=\; \exp\left[\frac{2i}{\hbar}\int d^4x d^4x' J^\dagger(x)\Delta_F(x - x')J(x')\right]$$

with $\Delta_F(x)$ defined in the same way as for real scalar fields.

1.B.(ii) Now show that we can write the full generating functional $\mathcal{Z}[J, J^\dagger]$ in terms of $\mathcal{Z}_o[J, J^\dagger]$ in the form

$$\mathcal{Z}[J, J^\dagger] \;=\; \exp\left[-\frac{i}{\hbar}\int d^4x \;\; V\left(\frac{-\hbar^2\delta^2}{\delta J(x)\delta J^\dagger(x)}\right)\right] \; \mathcal{Z}_o[J, J^\dagger]$$

where in the argument of $V(|\phi|^2)$ we have substituted $-\hbar^2\delta^2/\delta J(x)\delta J^\dagger(x)$ in place of $|\phi|^2$.

1.B.(iii) You can read off the diagram rules from the last expression. Assuming that $V(|\phi|) = g_o|\phi|^4/4!$, write down in 4-momentum space the 2 lowest-order graphs for the self-energy (2-point vertex part) for this theory.

QUESTION (2): MATSUBARA DIAGRAMS

2(A) Single Fermion Loops: We look at the simple non-relativistic Fermion Loop at finite T, in the Matsubara formalism; we will assume free fermions in a system that is isotropic and translationally invariant, so that the particle dispersion is just $\epsilon^o_{\mathbf{k}} = \hbar^2|\mathbf{k}|^2/2m$; and we write the Fermi distribution function as $f_{\mathbf{k}} = f(\epsilon^o_{\mathbf{k}})$

2.A.(i) The bare 'pair propagator' (NOT the pair polarization bubble!) for a non-relativistic fermionic system at finite temperature T is defined as $\chi_o(q) = \sum_k G_o(k)G_o(q-k)$, where $k = (\mathbf{k}, \epsilon)$ and $q = (\mathbf{q}, \omega)$ are 4-momenta. Show, by doing the Matsubara sum, and setting the external frequency argument to z, that

$$\chi_o(\mathbf{q}, z) \;=\; \sum_{\mathbf{k}} \frac{1 - f_{\mathbf{k}} - f_{\mathbf{q-k}}}{z - (\epsilon^o_{\mathbf{q-k}} + \epsilon^o_{\mathbf{k}})}$$

2.A.(ii) Now consider the polarization part, defined as $\pi_o(q) = \sum_k G_o(k+q)G_o(k)$ in the same notation as above, and given as usual in Matsubara notation by

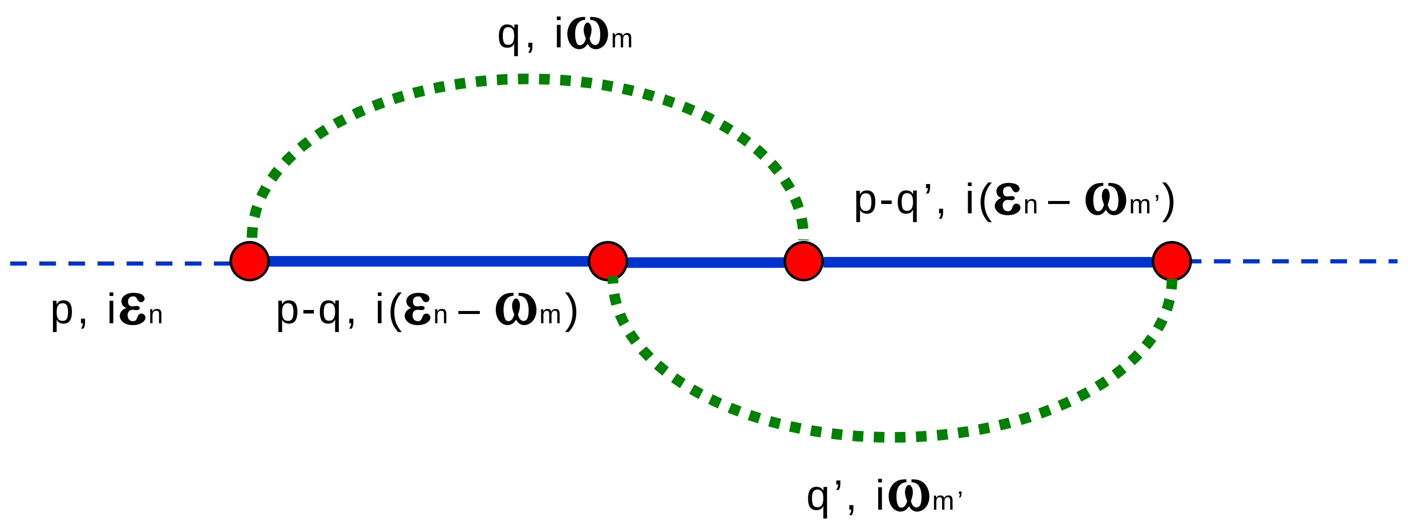
$$\pi_0(\mathbf{q}, i\omega_m) \quad = \quad \sum_{\mathbf{k}} \frac{f_{\mathbf{k}} - f_{\mathbf{k+q}}}{i\omega_m - (\epsilon^o_{\mathbf{k+q}} - \epsilon^o_{\mathbf{k}})}$$

We can actually do the momentum integral in 1 dimension. Carry this out, assuming that $T = 0$, so that the Fermi function $f(\epsilon) = \theta(\mu - \epsilon)$, where μ is the chemical potential. This should give you an analytic expression for $\pi_o(\mathbf{q}, i\omega_m)$.

2.A.(iii) Now a rather more lengthy calculation. Consider the lowest-order "crossed" graph for the electron-phonon self-energy $\Sigma_{\mathbf{p}}(i\epsilon_n)$, shown in the figure. Evaluate the frequency sums for this graph, to obtain an expression involving only integrations over the real-space momenta $\mathbf{q}$ and $\mathbf{q}'$.

See Figure on next Page

END of QUESTION SHEET 1



$\Sigma\,(p,\, i\varepsilon_n)$