Phys 508: HOMEWORK ASSIGNMENT No (4)

Thursday 26th March, 2015

DUE DATE: Wed Apr 15th 2015.

Assignments handed in late will not receive a full mark.

QUESTION (1): ANTIFERROMAGNETS

We consider an antiferromagnetic system in d spatial dimensions, with Hamiltonian

$$\mathcal{H} \;=\; -\frac{1}{2}K_2 \sum_j (\hat{S}_j^z)^2 \;-\; \sum_{ij} J_{<ij>} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j \tag{1}$$

with the sum $< ij >$ over nearest lattice neighbours, and $J_{ij} < 0$ in the exchange interaction.

1(i) Divide the system into 2 sublattices of spins, A and B, so that spins in A interact only with spins in B. Find the coupled equation of motion for spins $\mathbf{S}_i^A$ on sublattice A and spins $\mathbf{S}_j^B$ on sublattice B.

1(ii) Now assume that the spins have ordered with spins on sublattice A/B oriented in the $\pm\hat{z}$ direction. Treating the spins classically, and writing $\mathbf{S}_j = \pm S\hat{z} + \mathbf{s}_j$ for sublattice A/B, find the coupled equations of motion for the $\mathbf{s}_j$ on each sublattice. Then, Fourier transforming to functions

$$\mathbf{a}_\mathbf{q} = \sqrt{2/N} \sum_{i \in A} e^{-i\mathbf{q}\cdot\mathbf{r}_i} \mathbf{s}_i \qquad\qquad \mathbf{b}_\mathbf{q} = \sqrt{2/N} \sum_{j \in B} e^{-i\mathbf{q}\cdot\mathbf{r}_j} \mathbf{s}_j \tag{2}$$

find the coupled eqtns. of motion of the variables $\mathbf{m}_\mathbf{q}^{\pm}(t) = a_\mathbf{q}^{\pm} + b_\mathbf{q}^{\pm}$ and $\mathbf{n}_\mathbf{q}^{\pm}(t) = a_\mathbf{q}^{\pm} - b_\mathbf{q}^{\pm}$, where

$$a_\mathbf{q}^{\pm} = a_\mathbf{q}^x \pm i a_\mathbf{q}^y \qquad\qquad b_\mathbf{q}^{\pm} = b_\mathbf{q}^x \pm i b_\mathbf{q}^y \tag{3}$$

and from this derive the dispersion relation for the spin wave modes.

1(iii) The above analysis is inadequate for 2 reasons - it misses out the Berry phase, and it assumes long-range order where none may exist. Let us now write, for a 1-d lattice

$$\mathbf{S}_j(t) \;\;=\;\; S\alpha_j \, [1 - |\mathbf{m}_j(t)|^2]^{1/2} \, \mathbf{l}_j(t) \;+\; S\mathbf{m}_j(t) \tag{4}$$

where $\alpha_j = (-1)^j$, $|\mathbf{l}_j| = 1$, with the constraint $\mathbf{l}_j \cdot \mathbf{m}_j(t) = 0$, and we assume $|\mathbf{m}_j| \ll 1$.

Find the long-wavelength continuum form of the generating functional for this system, again using the Hamiltonian given above, in terms of an effective action $S_{eff}[\mathbf{l}(x,t), \mathbf{m}(x,t)]$; and demonstrate explicitly that the Berry phase term can be written in terms of the Pontryagin invariant.

QUESTION (2): IR BEHAVIOUR of GAUGE THEORIES

One can learn a great deal about the IR properties of 4-d QED by studying a simpler $2+1$ -dimensional model of non-relativistic fermions coupled to photons, with renormalized photon propagator

$$D_{ij}(q) \;=\; (\delta_{ij} - \hat{q}_i\hat{q}_j)D_\perp(q) \;=\; (\delta_{ij} - \hat{q}_i\hat{q}_j)\,\frac{1}{\chi|\mathbf{q}|^2 - i\gamma\omega/|\mathbf{q}|} \tag{5}$$

where $q = (\mathbf{q}, \omega)$, and i, j are spatial indices only - the electric part of the propagator is ignored. We write the fermionic propagator $\mathcal{G}(p, \epsilon)$ as

$$\mathcal{G}(p) \;=\; \int \mathcal{D}\mathbf{A}(q)\; e^{\frac{i}{2}\sum_q A_i(q) D^{-1}_{ij}(q) A_j(-q)}\; \bar{G}_p(q|\mathbf{A}) \tag{6}$$

where $\bar{G}_p(q|\mathbf{A})$ is a functional of a specific 'frozen' field configuration $\mathbf{A}(q)$ with components $A_i(q)$; its Fourier transform satisfies the differential equation

$$[i\partial_t - \frac{1}{2m}(-i\nabla - g_o\mathbf{A}(x))^2 + \mu]\;\mathcal{G}(x, x'|\mathbf{A}) \;=\; -\delta(x - x') \tag{7}$$

where q is the Fourier transform of $x - x'$, $x = (\mathbf{r}, t)$, etc., and g_o is a bare coupling.

2(i) Let us write $\bar{G}$ in 'proper time' representation as a WKB expansion of form

$$\bar{G}_p(x|A) \;=\; i\int_0^\infty ds e^{-isG_o^{-1}(p)} e^{i\Psi_p(s,x|A)} \;=\; i\int_0^\infty ds e^{-is(\epsilon - \varepsilon_\mathbf{p} + i\delta_p)} e^{i\sum_n g_o^n \psi_n(s;p,x|A)} \tag{8}$$

where ϵ and $\varepsilon_\mathbf{p}$ are measured from μ. Find a solution for $\psi_n(s; p, x|A)$ in recursive form, and show that

$$\psi_1(s; p, x|A) \;=\; \frac{g_o}{m}\sum_q e^{iqx}\,\frac{1 - e^{-is(\omega - \Omega_\mathbf{p}(\mathbf{q}))}}{\omega - \Omega_\mathbf{p}(\mathbf{q})}\;\mathbf{p}\cdot\mathbf{A}(x) \tag{9}$$

where $\Omega_\mathbf{p}(\mathbf{q}) = \varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_\mathbf{p}$.

2(ii) Now, by doing the functional integration over $\mathbf{A}(x)$, and using the approximation that $\Psi = \psi_1$, show that we can write $\mathcal{G}(p)$ in the form

$$\mathcal{G}(p) \;=\; i\int_0^\infty ds e^{-isG_o^{-1}(p)}\;\exp\left[i\frac{g^2}{m^2}\sum_q D_{ij}(q)p_ip_j\;\frac{1 - e^{-is(\omega - \Omega_\mathbf{p}(\mathbf{q}))}}{(\omega - \Omega_\mathbf{p}(\mathbf{q}))^2}\right] \tag{10}$$

2(iii) Finally, show that for a particle on a circular 2-d Fermi surface, but with finite frequency ϵ, we can do the integral over q to get the result in the IR limit:

$$\mathcal{G}_{p_F}(\epsilon) \;=\; i\int_0^\infty e^{is(\epsilon + i\delta) - \tilde{g}^2 (is)^{1/3}} \tag{11}$$

which can then be written in terms of Airy functions; here we define a renormalized coupling $\tilde{g}^2 = g_o^2[\Gamma(\frac{2}{3})p_F/8\pi^2\sqrt{3}(\chi^2\gamma)^{1/3}m]$.

END of QUESTION SHEET 4