QUICK NOTE ON FLUCTUATION DETERMINANTS

In the course notes a rather lengthy derivation is given of the fluctuation determinant for the path integral form of the Green function (section A.3.2 (a)). However this derivation is, while fairly general, also rather unwieldy and is not very practical.

Here is a much simpler and more intuitively obvious method. Suppose we have a Lagrangian of form

$$L(\varphi,\dot{\varphi};t) = \alpha(t)\dot{\varphi}^2 + \beta(t)\varphi\dot{\varphi} + \gamma(t)\varphi^2 + \eta(t)\dot{\varphi} + f(t)\varphi \qquad (1)$$

Now write the path $\varphi(t)$ as $\qquad \varphi(t) = \varphi_c(t) + q(t) \qquad (2)$

where $\varphi_c(t)$ is the "classical path" that minimizes the action, and $q(t)$ is some small deviation about this path. We now wish to evaluate the action $S[\varphi]$ up to quadratic order in the deviations $q(t)$, $\dot{q}(t)$. We have

$$\left.\begin{aligned} S[\varphi,\dot{\varphi}] &= S[\varphi_c+q,\ \dot{\varphi}_c+\dot{q}] \\ &= \int_{t_1}^{t_2} dt \Big\{ \alpha(t)[\dot{\varphi}_c^2 + 2\dot{\varphi}_c\dot{q} + \dot{q}^2] + \beta(t)[\varphi_c\dot{\varphi}_c + \varphi_c\dot{q} + q\dot{\varphi}_c + q\dot{q}] \\ &\qquad + \gamma(t)[\varphi_c^2 + 2\varphi_c q + q^2] + \eta(t)[\dot{\varphi}_c + \dot{q}] + f(t)[\varphi_c + q] \Big\} \end{aligned}\right\} (3)$$

from which all terms linear in q and $\dot{q}$ must vanish in the integral, since $S[\varphi_c,\dot{\varphi}_c]$ minimizes this integral - we thus get

$$S[\varphi,\dot{\varphi}] = S_c[\varphi_c,\dot{\varphi}_c] + \int_{t_1}^{t_2} dt\,[\alpha(t)\dot{q}^2 + \beta(t)q\dot{q} + \gamma(t)q^2]\Big|_0^0 \qquad (4)$$

where $\int_0^0$ indicates that $\qquad q(t_1) = q(t_2) = 0 \qquad (5)$

by hypothesis (since $\delta\varphi(t_1) = \delta\varphi(t_2) = 0$ in the original variational problem).

Turning now to the path integral form for $G(\varphi_2,\varphi_1;t_2,t_1)$, we see that we can write

$$\left.\begin{aligned} G(\varphi_2,\varphi_1;t_2,t_1) &= \int_{\varphi_1(t_1)}^{\varphi_2(t_2)} \mathcal{D}\varphi(t)\, e^{\frac{i}{\hbar}S[\varphi,\dot{\varphi}]} \\ &\longrightarrow A(t_2-t_1)\, e^{\frac{i}{\hbar}S_c[\varphi_2,\varphi_1]} \end{aligned}\right\} (6)$$

where from (4):

$$A(t_2-t_1) = \int_{q(t_1)=0}^{q(t_2)=0} \mathcal{D}q(t)\, \exp\Big\{\frac{i}{\hbar}\int_{t_1}^{t_2} dt\,[\alpha(t)\dot{q}^2 + \beta(t)q\dot{q} + \gamma(t)q^2]\Big\} \qquad (7)$$

We see that the result for $A(t)$ is completely independent of $\eta(t)$ and $f(t)$.

Consider now the example of a simple harmonic oscillator, with Lagrangian

$$L_0(\varphi,\dot{\varphi}) = \tfrac{1}{2}m(\dot{\varphi}^2 - \omega_0^2\varphi^2) \tag{8}$$

ie., we drop $\beta(t)$ in (1), and let $\alpha(t)$ and $\gamma(t)$ be constants. Then the fluctuation determinant $A(t_2-t_1)$ is just

$$A(t_2-t_1) \equiv A(T) = \int_0^0 \mathcal{D}q(t)\, \exp\left\{\frac{i}{\hbar}\int_0^T dt\, \frac{M}{2}(\dot{q}^2 - \omega_0^2 q^2)\right\} \tag{9}$$

where we define $T = t_2 - t_1$, noting that the result for A is translationally invariant in time.

To evaluate this, we use the condition that $q(0) = q(T) = 0$, and write $q(t)$ as a Fourier sum:

$$q(t) = \sum_{n=1}^{\infty} q_n \sin\left(\frac{\pi n t}{T}\right) \tag{10}$$

(there is no $n=0$ term - it would be zero). We then have

$$\left.\begin{aligned} \int_0^T dt\, q^2(t) &= \frac{T}{2}\sum_n q_n^2 \\ \int_0^T dt\, \dot{q}^2(t) &= \frac{\pi^2}{T^2}\sum_{n_1 n_2} n_1 n_2 \int_0^T dt\, \cos\left(\frac{n_1 \pi t}{T}\right)\cos\left(\frac{n_2\pi t}{T}\right) = \frac{T}{2}\sum_n \left(\frac{n\pi}{T}\right)^2 q_n^2 \end{aligned}\right\} \tag{11}$$

so that $$A(T) \propto \lim_{N\to\infty} \prod_{n=1}^{N} \int_{-\infty}^{\infty} dq_n\, \exp\left\{\frac{im}{2\hbar}\sum_n \left[\left(\frac{n\pi}{T}\right)^2 - \omega_0^2\right] q_n^2\right\} \tag{12}$$

where we have made the sum well-defined by truncating it to a finite number N, before taking the limit $N \to \infty$. We will not need to deal with the N-dependent normalisation factor here - this is why I simply write that the L.H.S. is proportional to the R.H.S. The integrals are simple Gaussians, so that

$$A(T) \propto \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2} - \omega_0^2\right)^{-1/2} \propto \left(\frac{\sin\omega_0 T}{\omega_0 T}\right)^{-1/2} \tag{13}$$

Now we can fix the constant of proportionality by noting that when $\omega_0 = 0$, we should get the easily computed free particle result:

$$A(T) \xrightarrow[\omega_0=0]{} \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} \tag{14}$$

Thus we have, for the SHO: $$A(T) = \left(\frac{m\omega_0}{2\pi i\hbar \sin\omega_0 T}\right)^{1/2} \tag{15}$$

Now let's look at this in a slightly more general way. There is no particular reason to resolve the fluctuation path $q(t)$ into Fourier components — any orthonormal & complete set of functions will do. Thus let us write

$$Q(t) = \left[Q_c(t) + \sum_n q_n \chi_n(t)\right] = Q_c(t) + q(t) \tag{16}$$

where

$$\left.\begin{aligned} \int_{t_1}^{t_2} dt\, \chi_n(t)\,\chi_{n'}(t) &= \delta_{nn'} \\ \chi_n(t_2) = \chi_n(t_1) &= 0 \end{aligned}\right\} \tag{17}$$

Now, assuming there is only one classical path $Q_c(t)$ (ie., one path which extremizes the action), so that we do not have to worry about any interference between paths, let's consider a form

$$L = \frac{m}{2}\dot{Q}^2 - V(Q) \tag{18}$$

which generalizes the harmonic oscillator to more general potentials, but which avoids the terms in (1) which vary in time (and also ignores terms linear in $\dot{Q}$). Now we know that the classical path satisfies Lagrange's eqtn., viz., that

$$\frac{\delta S}{\delta Q_c} = m\ddot{Q}_c + V'(Q_c) = 0 \tag{19}$$

(vanishing of the first variational derivative); and by the same manoeuvres as before, we choose the orthonormal functions, like $q(t)$, to satisfy the eigenvalue eqtn:

$$\frac{m}{2}\ddot{\chi}(t) + V''(Q_c)\chi_n(t) = -\omega_n \chi_n(t) \tag{20}$$

obtained from the 2nd variational derivative of S around the path $Q_c(t)$.

It then follows that the fluctuational prefactor can be written as

$$\left.\begin{aligned} A(t_2-t_1) &= \int_{q(t_1)=0}^{q(t_2)=0} \mathcal{D}q(t)\, \exp\left\{\frac{i}{\hbar}\int dt\, \frac{1}{2}\frac{\delta^2 S}{\delta Q^2}\Big|_{Q_c} q^2(t)\right\} \\ &= \int_0^0 \mathcal{D}q(t)\, \exp\left\{\frac{i}{\hbar}\int_{t_1}^{t_2} dt\, L_2(q,\dot{q})\right\} \end{aligned}\right\} \tag{21}$$

where the "fluctuational Lagrangian" is just $L_2(q,\dot{q}) = \frac{m}{2}\ddot{q} - V''(Q_c)q^2$ (22)

But then we have

$$\left.\begin{aligned} A(t_2-t_1) &= \int \mathcal{D}q(t)\, \exp\left\{\frac{-i}{2\hbar}\int dt\left[m\frac{d^2}{dt^2} + V''(Q_c)\right]q^2\right\} \\ &\propto \lim_{N\to\infty}\prod_{n=1}^{N}\int d\chi_n\, \exp\left\{\frac{-i}{2\hbar}\int dt\left(m\frac{d^2}{dt^2} + V''(Q_c)\right]\chi_n^2\right\} \\ &\longrightarrow \frac{1}{\left[\det\left(-m\frac{d^2}{dt^2} - V''(Q_c)\right)\right]^{1/2}} = \prod_{n=1}^{N}\frac{1}{\omega_n^{1/2}} \end{aligned}\right\}$$