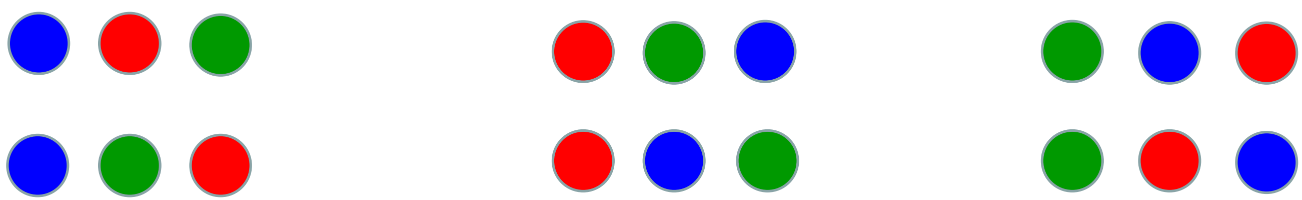
LECTURES 1 AND 2 (Mon 11 Jan, Wed 13 Jan 2021)
(with some revision on Mon 18 Jan 2021)

PROBABILITY

N Distinguishable objects: the definition is obvious. There are N! different possible arrangements (permutations)

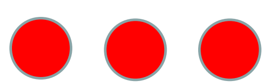
The object in the first position on the line may be chosen in N different ways, that in the second position in N−1 ways, and so on. The number of possible arrangements is therefore

$$N(N-1)(N-2)(1) = N!$$



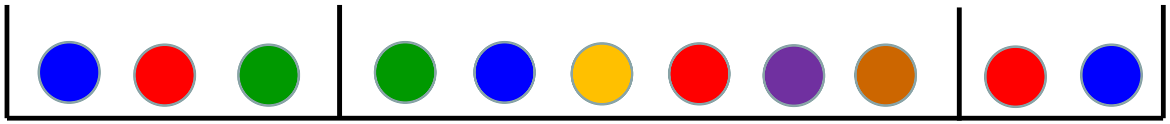
3! = 6

N Indistinguishable objects: the definition is absolute – there is no difference in principle between the different objects (ie., it is not just our inability to distinguish them). There is only ONE different possible arrangements (permutation)



MULTINOMIAL COMBINATORICS

Here we partition the N objects into m different groups. We want to know the number of permutations/outcomes in which internal permutations of objects inside a box are considered to be indistinguishable



n_1 objects ($n_1 = 3$)

n_2 objects ($n_2 = 6$)

n_3 objects ($n_3 = 2$)

By the same arguments as before – we have a total of

$$C^N_{\{n_j\}} \equiv C^N_{n_1,n_2,..n_m} = \frac{N!}{n_1!n_2!....n_m!}$$ permutations

Example – a set of particles with spins s_k such that $m_k = 2s_k + 1$

since the k-th group of identical objects can be rearranged in n_k! ways without changing anything, and we can do this for any of the m different sub-groups.

Special case: $$C^N_n = \frac{N!}{n!(N-n)!}$$

Binomial distribution (cf toin cosses, or spin-1/2 particles)

This is the number of ways we can get n heads if we toss the coin N times

DISCRETE PROBABILITIES for N objects

We've found the number of permutations – now we must assign a probability to each one.

The simplest is to assign equal probabilities to each permutation or "outcome". If the total number of permutations is X, the probability of each outcome is 1/X

Example: Coin tossing

We look for the probability of getting n heads (ie., spin up) from N tosses. The total number of possible outcomes is 2^N; each therefore has probability $1/2^N$

We then have
$$P_N(n) = \frac{C_n^N}{2^N} = \frac{1}{2^N}\left(\frac{N!}{n!(N-n)!}\right)$$

Unequal Probabilities: Suppose the probability of getting a head (spin up) is p_+, and that of getting a down is $p_- = 1 - p_+$. Then we have

$$P_N(n) = C_n^N p_+^n p_-^{N-n)} \equiv \left(\frac{N!}{n!(N-n)!}\right) p_+^n (1-p_+)^{(N-n)}$$

because the probability of getting any one of the combinations with n heads and N−n tails is, by assumption, just the product over the probabilities for each throw. When $p_+ = 1/2$, this just reduces to the previous result.

MULTINOMIAL PROBABILITIES

Suppose we have N identical balls which we distribute in m different cells or boxes, but now the probability of going into the k-th box is p_k, where k= 1;2;$\cdots$m (and where of course $\sum_k p_k = 1$). As we saw before, the number of different ways of doing this is just the multinomial coefficient

However now the weighting attached to any one of these ways is: $\prod_k (p_k)^{n_k}$

It then follows that the probability $P_N(n_1; n_2;\cdots n_m)$ of getting an outcome in which there are n_k balls in the k-th box is just

$$
\begin{aligned}
P_N(n_1, n_2, \cdots n_m) &= C^N_{n_1,n_2,..n_m} \prod_{k=1}^{m} p_k^{n_k} \\
&= \delta(N - \sum_k n_k) \left(\frac{N!}{n_1! n_2! n_m!}\right) p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}
\end{aligned}
$$

Kronecker Delta Function

EXAMPLE OF BINOMIAL DISTRIBUTION

Example : A card game.

Suppose we are dealt 7 cards from a 52-card pack. What is the probability that this hand contains 3 Aces?

To do this we need to first ask how many possible outcomes there are for the 7 cards that are dealt; we then ask how many of these give 3 Aces.

(1) Total number of possible distinguishable arrangements is the binomial $C^{52}_{7} \equiv C^{52}_{45} = 52!/(7!\ 45!)$ This is because we can re-order the first 7 cards 7! times, and the last 45 cards 45! times

(2) To find how many of these are Aces, we note first that it does not matter which Aces we get. We need to multiply the number of ways of getting 3 of the 4 Aces (without caring which ones), by the total number of outcomes for the other 4 cards that are dealt, with the constraint that these other cards are NOT Aces. The first number is $C^{4}_{3} = 4$.To find the second number, we note that there are 48 cards that are not Aces, and we are getting 4 of these. So this latter number is $C^{48}_{4} = 48!/(44!\ 4!)$

The final result for the probability $P^{\{7\}}_{AAA}$ is then

$$P^{\{7\}}_{AAA} = \frac{C^4_3 C^{48}_4}{C^{52}_7} = 4 \times \frac{48!}{4!\ 44!} \times \frac{7!\ 45!}{52!} = 7.6.5.4 \frac{45}{52.51.50.49}$$

which if we work it out gives $P^{\{7\}}_{AAA} \sim 0:00582$, ie., roughly 1/172.

EXAMPLE OF MULTINOMIAL DISTRIBUTION

Example : Another card game.

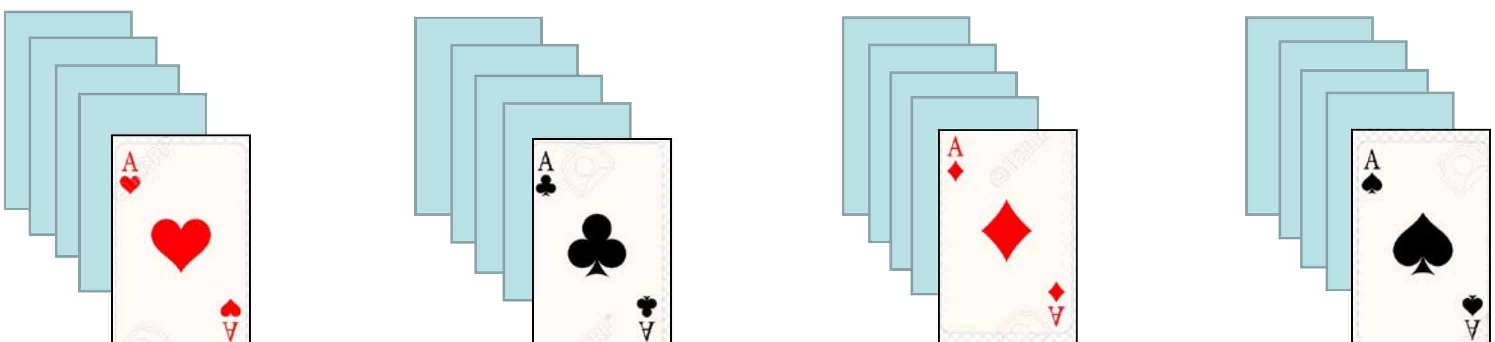
Suppose we have 4 players, each on is dealt 5 cards. What is the probability that each one of them has exactly one Ace?

This generalizes last problem to a multinomial distribution. We must first ask how many possible outcomes there are for the 4 batches of 5 cards that are dealt; we then ask how many of these give 1 Ace in each hand.

(1) There are $C^{52}_{5:5:5:5:32} \equiv 52!/[(5!)^4 32!]$ ways of distributing the cards amongst 4 hands of 5 cards, and then amongst the remaining 32 cards.

(2) There are 4! ways of ordering the 4 Aces. There are then 48 cards left, that are not Aces - these can be dealt out to the 4 different hands in a total of $C^{48}_{4:4:4:4:32} \equiv 48!/[(4!)^4 32!]$ times.

Probability is then $$P_{4A}^{\{4\times 5\}} = \frac{4! \times C^{48}_{4.4.4.4.32}}{C^{52}_{5.5.5.5.32}} = \frac{5^4 \times 24}{52.51.50.49} \sim 2{:}31\times10^{-3} \sim 1/433$$



See Notes and Homework assignments for other examples