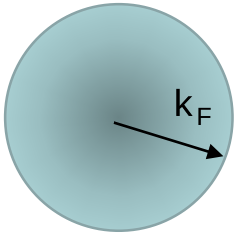
FERMI LIQUIDS

Recall that for a Fermi gas, we found that by relating the number N of particles to the chemical potential, via

$$N = \int_0^\infty V g(E) f(E, \mu, T) dE \quad = \quad V\rho$$



we got:

T=0 FERMI GAS: Then we just had $\rho = \int_0^{E_F} g(E) dE$

so that $N = V \int_o^{E_F} g(E) dE \rightarrow \frac{V}{3\pi^2}\left(\frac{2mE_F}{\hbar^2}\right)^{3/2}$ where $E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3} = (\hbar k_F)^2 / 2m$;

In what follows we will

(i) study the "degenerate Fermi regime", ie., the regime where $kT \ll E_F$;

(ii) discuss how the results generalize to systems with interactions between the fermions

(iii) Look at some real physical examples, including metals, 3He liquid, and stars.

(1) DEGENERATE REGIME; LOW-T EXPANSIONS

Suppose I have a function $I = \int_0^\infty H(E) f(E) dE$ which I want to find for $kT \ll E_F$. Then I integrate by parts:

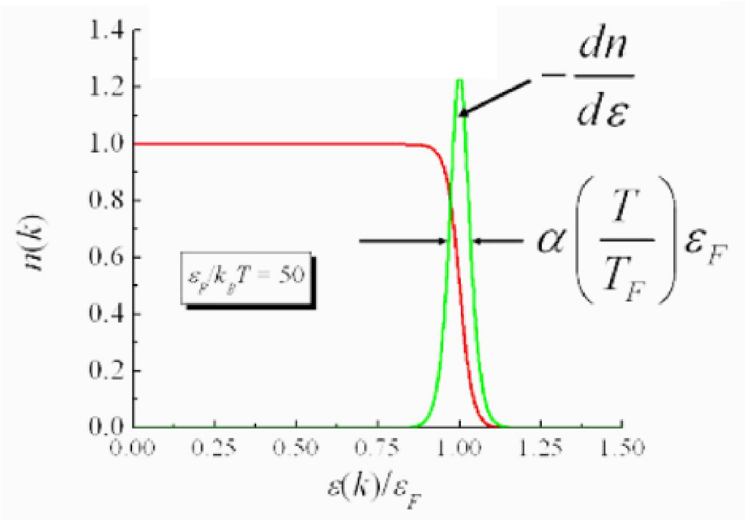
$$I = \int_0^\infty h'(E) f(E) dE = \left[f(E) h(E) \right]_0^\infty - \int_0^\infty f'(E) h(E) dE \rightarrow -\int_0^\infty f'(E) h(E) dE$$

Where $h(E) \equiv \int_o^E H(\varepsilon) d\varepsilon$, ie. $h'(E) = H(E)$ and $\left[f(E) h(E) \right]_0^\infty = 0$ since $f(\infty) = h(0) = 0$.

Now let's look at this function. We have

$$I = -\int_0^\infty f'(E)h(E)dE$$

and the key point here is that df/dE is very sharply peaked around the Fermi energy when $kT \ll E_F$; it acts as a "window function", which tends to a delta-function when $T \rightarrow 0$.



Derivative dn/dε of the Fermi-Dirac function, in green. It has a width α roughly equal to $6k_BT$. It is shown here for $k_BT/E_F = 0.02$

To do the integral systematically we make a Taylor-MacLaurin expansion (often called in this context a Sommerfeld expansion) of the integral, ie., we write:

$$h(E) = h(\mu) + (E-\mu)h'(\mu) + \frac{1}{2}(E-\mu)^2 h''(\mu) + \ldots$$

so

$$I = h(\mu)I_0 + h'(\mu)I_1 + h''(\mu)I_2 + \ldots$$

where $I_0 = -\int_0^\infty f' dE = 1$ and $I_1 = -\int_0^\infty (E-\mu)f' dE \sim -\int_{-\infty}^\infty (E-\mu)f' dE = 0$

& also $I_2 = -\frac{1}{2}\int_0^\infty (E-\mu)^2 f' dE \sim -\frac{1}{2}\int_{-\infty}^\infty (E-\mu)^2 f' dE = \frac{1}{2}(k_BT)^2 \int_{-\infty}^\infty \frac{x^2 e^x}{(1+e^x)^2}dx = \frac{\pi^2}{6}(k_BT)^2$

Thus we finally get: $I = h(\mu) + \frac{\pi^2}{6}(k_BT)^2 h''(\mu) + \text{etc.}$

In our case, the function h(E) is just $h(E) = \int_0^E g(\varepsilon)d\varepsilon = \frac{2}{3}cE^{3/2}$ where $c = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$.

(a) Sommerfeld Expansion for Chemical Potential: Let's re-evaluate the integral determining the chemical potential for a given number N of particles. We have again that

$$\frac{N}{V} = \int_0^\infty f(E)g(E)dE \quad = \quad -\int_0^\infty f'(E)h(E)dE$$

where $h(E) = \int_0^E g(\varepsilon)d\varepsilon \;=\; \frac{2}{3}cE^{3/2}$

so that $h'(E) = cE^{1/2} = g(E)$ and $h''(E) = \frac{1}{2}cE^{-1/2} = \frac{1}{2}\frac{g(E)}{E}$

and we also established that $I \;=\; \frac{N}{V} \;=\; h(\mu) + \frac{\pi^2}{6}(k_BT)^2 h''(\mu) \;+\; \text{etc.}$

& so we get $\frac{N}{V} \;=\; \int_0^\mu g(\varepsilon)d\varepsilon + \frac{\pi^2}{12}(k_BT)^2\frac{g(\mu)}{\mu}$

To get an answer we now need to evaluate $\int_0^\mu g(\varepsilon)d\varepsilon$ which we do by noting that

$\int_o^{E_F} g(E)dE \;=\; \int_0^\mu g(E)dE + \frac{\pi^2}{12}(k_BT)^2\frac{g(\mu)}{\mu}$ & noting that in this expansion, $E_F - \mu \ll E_F$.

We then expand $\int_\mu^{E_F} g(E)dE = \frac{\pi^2}{12}(k_BT)^2\frac{g(\mu)}{\mu}$

and then since g(E) varies slowly with energy compared to the window function df/dE, we have

$(E_F - \mu)g(E_F) = \frac{\pi^2}{12}(k_BT)^2\frac{g(E_F)}{E_F}$ so that finally $\mu = E_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2\right] \;+\; O(k_BT)^4$

(b) Some other low-T quantities: one uses the same technique to find other low-T quantities, looking for the small deviations from the T=0 behaviour. Generally speaking the algebraic manoeuvres are rather tedious, and one doesn't learn a lot from them, but I will summarize what one does here for the energy and specific heat.

Energy U: We add a factor of E in the integration: $U = V\int_0^\infty E g(E) f(E) dE$

Integrating again by parts, we get $h(E) \equiv \int_0^E \varepsilon g(\varepsilon) d\varepsilon = c\int_0^E \varepsilon^{3/2} d\varepsilon = \frac{2}{5} c E^{5/2}$

so that $h'(E) = cE^{3/2}$ and $h''(E) = \frac{3}{2} c E^{1/2}$

Now comes the tedious part. We expand again the integrations, and find (see the course notes for details):

$$U = \frac{3}{5} N k_B T_F + \frac{\pi^2}{4} N \frac{(k_B T)^2}{k_B T_F}$$

ie., $U = U_0[1 + \Delta U(T)]$, with:

$$U_0 = \frac{3}{5} N E_F$$
$$\Delta U(T) \sim O\left(\frac{T}{T_F}\right)^2$$

Thus the T=0 energy is big; $U_0 \sim O(NT_F)$; whereas $\Delta U(T)$ is a small correction. The specific heat, which measures the number of "active" degrees of freedom, is also small, since it only comes from the T^2 term:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\pi^2}{2} N k_B \frac{T}{T_F} = \gamma T \sim N(T/T_F)$$

(cf. the classical high-T result: $C_V = \frac{3}{2} N k_B$)

Pressure p: We have $p = -\left(\frac{\partial F}{\partial V}\right)_T$ so that $p_0 = -\left(\frac{\partial U_0}{\partial V}\right)_T = \frac{2}{5}\frac{N}{V} E_F$

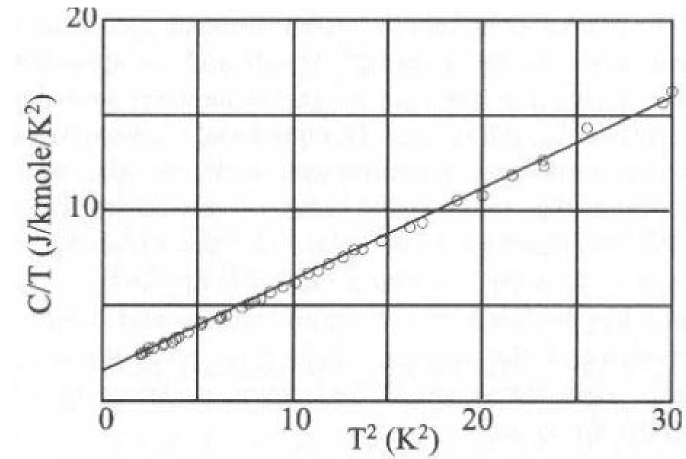
This is a v big pressure, with small T^2 correction. Note that $\frac{p_0 V}{U_0} = \frac{2}{3}$

EXAMPLE 1: METALS Metals are all around us – they are very good examples of degenerate Fermi liquids. This is because one has $E_F \sim 2\text{-}8$ eV, depending on the electron density. Since 1 eV ~ 11,604K, then at room temperature (~ 300K) we have $T/T_F < 10^{-2}$.

We might then expect a linear specific heat. In fact one finds a form that can be fit, at low T, to the form

$$C_V(T) = \gamma T + \beta T^3$$

as we see from the graph at right, for Cu metal. The reason for the T^3 term is acoustic phonons, which like photons have a linear dispersion and hence an energy ~ T^4 in 3 dimensions.



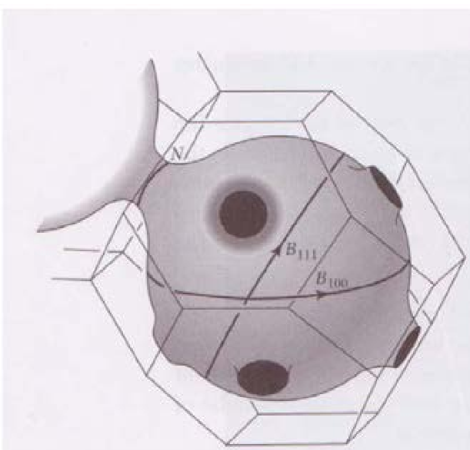
Plot of $C_V(T)/T$ as a function of T^2 for *Cu* metal

Notice that the linear term actually tells us the density of states g(E) around the Fermi energy E_F. To see this for the Fermi gas, we note that since we have

$$\gamma T = \frac{\pi^2}{2} N k_B \frac{T}{T_F}$$ we can also then write $$\gamma = V\frac{\pi^2}{3} g(E_F) k_B^2$$ by simple algebra

Finally, note that the actual shape of the momentum space Fermi surface in metals is strongly modified by the periodic potential of the ions (if we are dealing with a crystal lattice – and disordered systems are not usually conductors).

This is a messy subject, but is extremely important when one is looking at real metals. So is the effect of electron-electron interactions.....

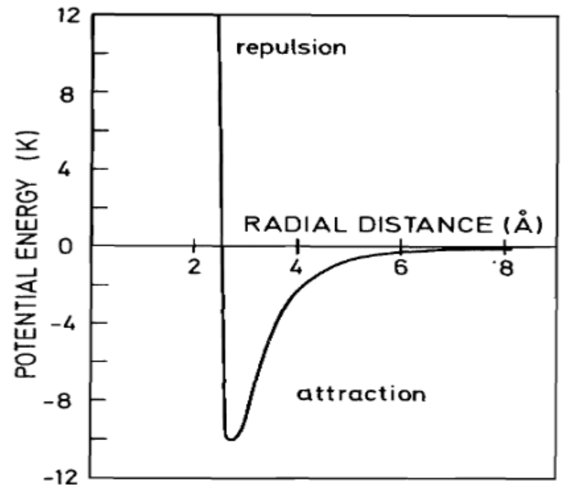


Fermi surface of Cu metal

EXAMPLE 2: NORMAL LIQUID ^3He

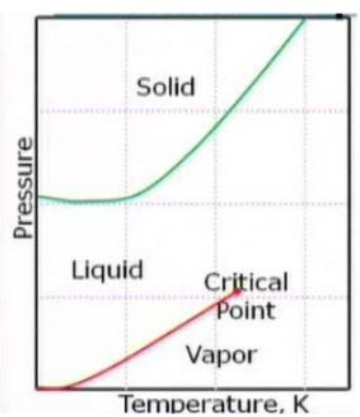
Here the fermions are the ^3He atoms (some 6000 times more massive than an electron). The Fermi energy E_F is thus much lower – we expect it to be E_F ~ 2-3 K. Historically this system is interesting, because it was the system whose study, in the hands of Landau, which opened the way to our modern understanding of the effect of interactions.

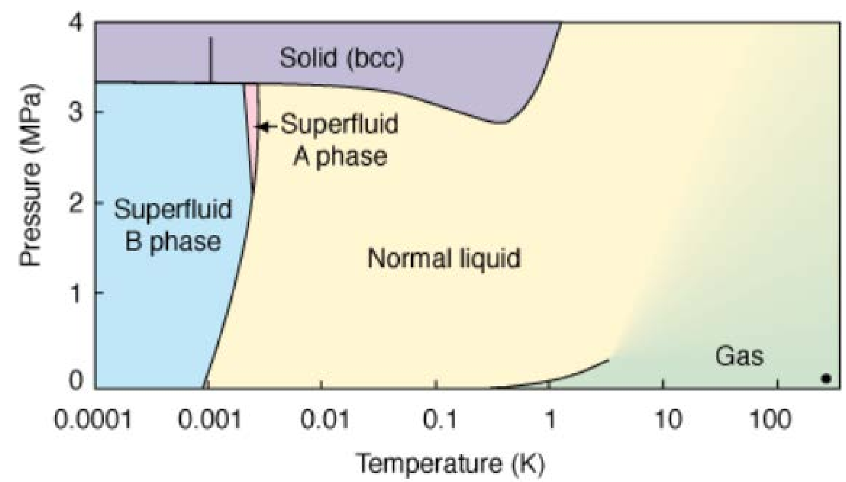
Some facts about 3He: The first important question is – what do the interactions between the atoms look like? The effective potential is shown at right – it has a short-range repulsion coming from the Pauli exclusion force, and a long-range van der Waals attraction. The energy scale here is a few K. The system is liquid because of the van der Waals attraction; the zero-point motion in the potential well is large.



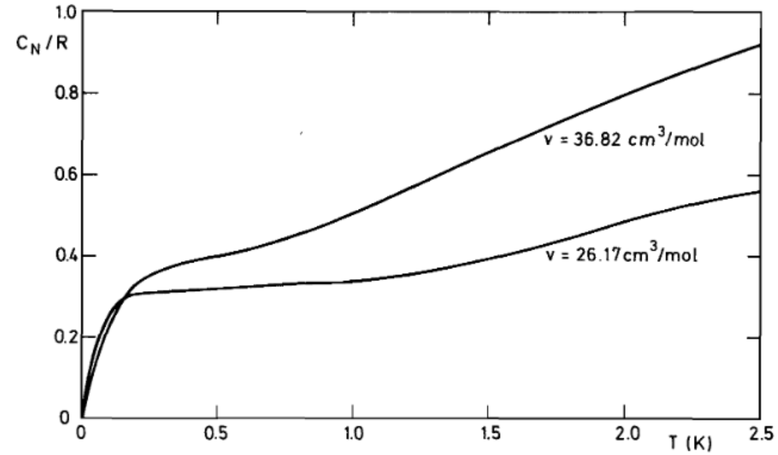
Potential energy between two 3He atoms

The phase diagram is shown below. At high T it looks similar to 4He, without the Bose-condensation of the 4He atoms. However at much lower T we can get superfluidity through the attractive interaction between the atoms, which causes Cooper pairing, just as in superconductors. There are even more phases than this, which are revealed by applying a magnetic field (and the solid also has multiple magnetic phases)



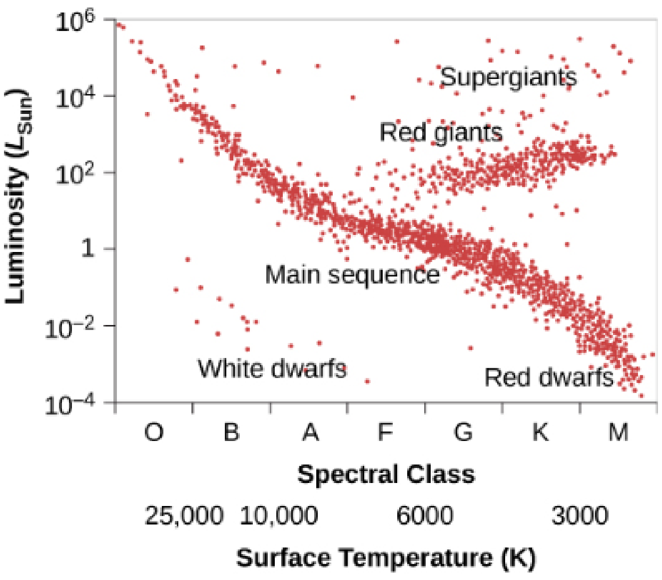


The specific heat of ^{3}He is very interesting; it does not behave linearly at all. The reason for this, and a fairly complete discussion of the properties of interacting fermions, was given in Landau's "theory of Fermi liquids" (1956-59). For temperature above ~ 0.3K the system behaves much like a gas. But below 0.3K the interactions cause the "quasiparticle Excitations' to have an effective mass much larger than the bare ^{3}He mass (and so a larger specific heat coefficient). The quasiparticle however are then weakly-interacting fermions, and so we can understand everything in terms of them and their weak residual interactions.



The specific heat of liquid ^{3}He, taken for 2 different molar volumes, corresponding to pressures P = 0 atm and P = 30 atm respectively.

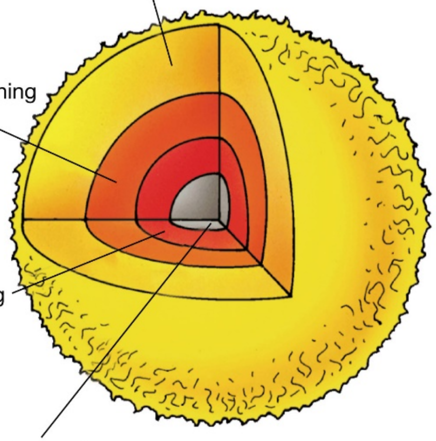
EXAMPLE 3: STARS – from WHITE DWARFS to NEUTRON STARS: In stars we deal principally with electrons, photons, plus various nucleons. We can classify them using the Hertzprung-Russell diagram; the theory behind this allows us to understand them very well.

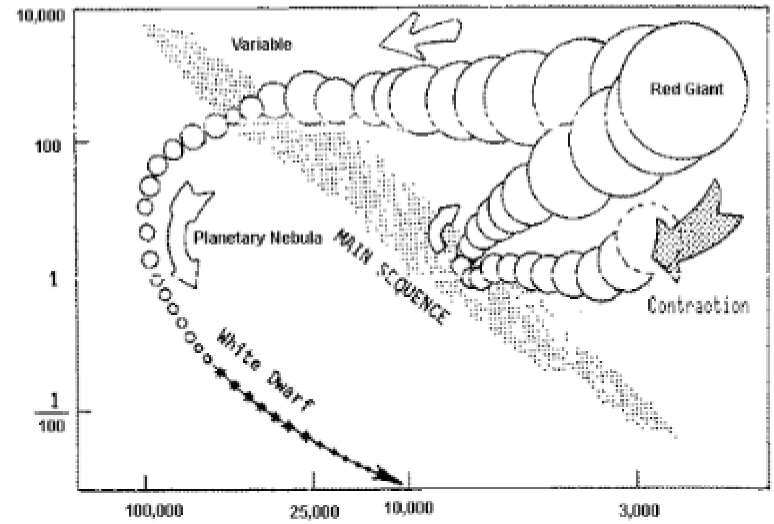


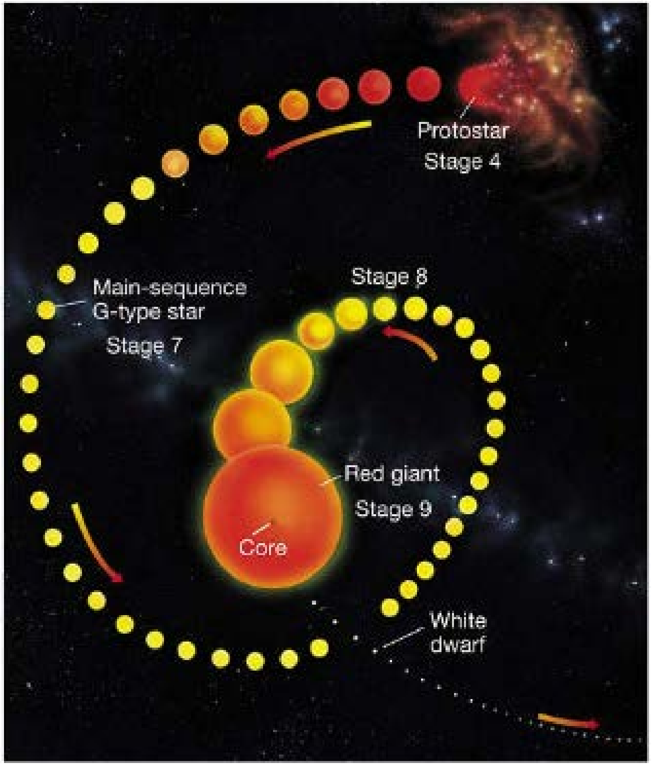
Stars can be viewed as gas spheres, with the outward radiation pressure balancing the inward gravitational attraction, & luminosity $L \sim O(R^2T^4)$, where R is the stellar radius.

The ZOOLOGY of STARS

1. LOW-MASS STARS: When $0.08\ M_S < M < 8M_S$, where M_S is the solar mass, we get the life-cycle shown. The balance between gravity and radiation pressure depends on the total mass of the star. The initial mass of the star is controlled by the collapse of the protostellar nebula to the main sequence. The initial position of the star on the main sequence is controlled by this mass.





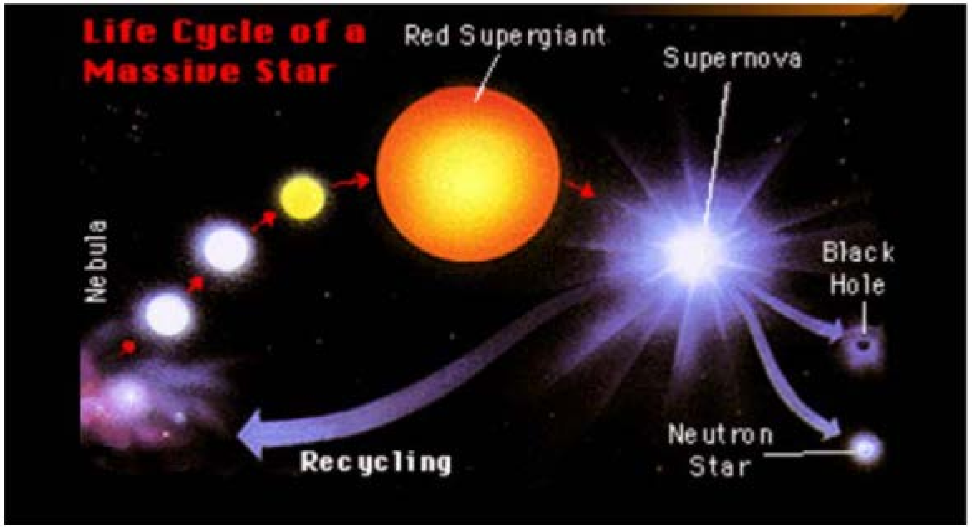


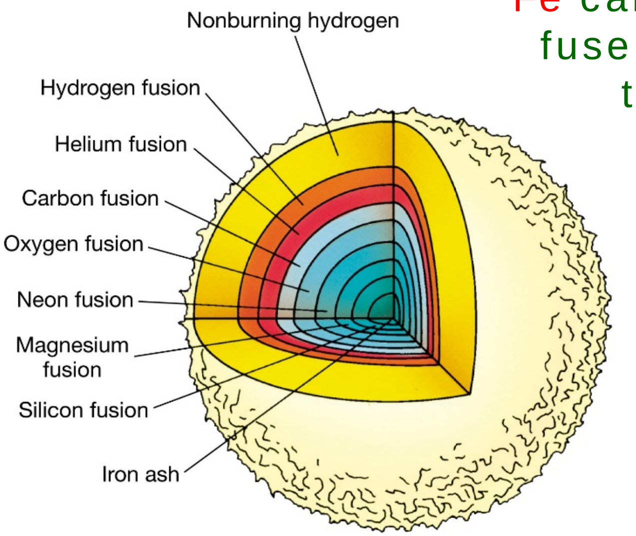
The rate of H → He fusion increases very rapidly with core temperature T; and T increases with mass. At higher T one can go to He → C burning, which goes even faster. Thus more massive stars are much more luminous, and have much shorter life spans.

Main sequence stars, with mass < $8M_S$, never get past C; as they run out of He, the cores collapse slowly, and the star balloons out to red giant form. When the H and He are exhausted, it slowly collapses to a white dwarf.

TOP: Evolution of a star passing via the main sequence
BELOW: Evolution of the sun in 250 million year increments

1. HIGH-MASS STARS: When $8M_S < M < 100M_S$, the stellar life-cycle is quite different. The mass is now high enough to cause core compression to densities and temperatures where C nuclei may fuse to O, then Ne, and so on up to Fe; each state requires an increase in T. As time goes by the star acquires an onion structure. Note that Fe can't further fuse – this is the end of the line.





Burning up the Si takes only a day or so; now suddenly out of fuel, the star instantly collapses, with the central regions exceeding nuclear densities. The enormous gravitational energy is converted to fission, plus neutrinos. A shock wave 'rebound' blows out the infalling matter, and the result is a supernova, with peak luminosity ~ 10^{10} L_S . This seeds the galaxy with heavy elements – all such elements come from supernova explosions.

The core remnant either collapses to a neutron star, made largely of neutrons and protons, or, if its mass exceed roughly 2.4 M_S, to a black hole. Black holes can also form by other processes – their masses can reach > 40×10^9 M_S, in galactic cores


M1, the 'Crab Nebula' supernova remnant

PHYSICS of WHITE DWARFS

Once a low-mass star has exhausted its fuel, it is held up against gravity by the electron Fermi degeneracy pressure. To find its structure let's balance these.

(i) Gravitational energy: to find this we successively remove shells at radius r of thickness dr; the energy required is

$$dE = \frac{G(\rho 4\pi r^2 dr)\left(\rho \frac{4}{3}\pi r^3\right)}{r} = G\frac{(\rho 4\pi)^2}{3} r^4 dr$$

to give

$$E = \int_R^0 dE = G\frac{(\rho 4\pi)^2}{3}\int_R^0 r^4 dr = G\frac{(\rho 4\pi)^2}{3}\frac{1}{5}R^5 = \frac{3G}{5R}\left(\rho\frac{4}{3}\pi R^3\right)^2 = \frac{3}{5}\frac{GM^2}{R}$$

For a star of radius R

to give a total gravitational energy $U_g = -\frac{3}{5}\frac{GM^2}{R} = -C_g\frac{GM^2}{R}$

(ii) Degeneracy Energy: As we know the degeneracy energy at T=0 is just

$$U_e = \frac{3}{5}NE_F = \frac{3}{5}N\frac{\hbar^2}{2m_e}\left(3\pi^2\frac{N}{V}\right)^{2/3} = \frac{3^{5/3}\hbar^2\pi^{4/3}}{10m_e}\frac{N^{5/3}}{V^{2/3}}$$

To find N we suppose a star is made from an equal mixture of protons & nucleons, so that

$$U_e = C_e\frac{M^{5/3}}{R^2} \quad \text{where} \quad C_e = \frac{3^{5/3}\hbar^2\pi^{4/3}}{10m_e(2m_p)^{5/3}\left(\frac{4}{3}\pi\right)^{2/3}}$$

(iii) Total Energy: we now get

$$U = U_g + U_e = -C_g G\frac{M^2}{R} + C_e\frac{M^{5/3}}{R^2}$$

& stability requires $\frac{dU}{dR} = C_g G\frac{M^2}{R^2} - 2C_e\frac{M^{5/3}}{R^3} = 0$

Thus we finally get

$$\boxed{R_{eq} = \frac{2C_e}{C_g G}M^{-1/3}}$$

Chandrasekhar's relativistic modification: the 19-yr old Chandrasekhar realized the previous argument was wrong, because the electrons at this density were at relativistic energies. If one now writes

$$E_F = \sqrt{(\hbar c k_F)^2 + (m_e c^2)^2} \quad \sim \quad \hbar c k_F \quad = \quad \hbar c\left(3\pi^2 \frac{N}{V}\right)^{1/3}$$

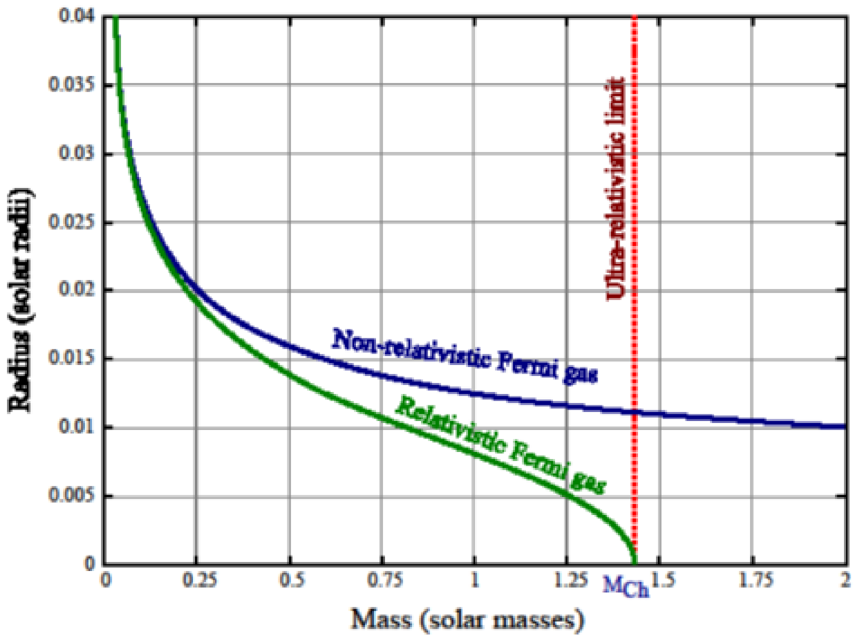
then the degeneracy energy becomes $U_e = C_2 \frac{M^{4/3}}{R}$

& the stability criterion becomes $\frac{dU}{dR} \quad = \quad \frac{1}{R^2}(C_g G M^2 - C_2 M^{4/3}) \quad = \ 0$

This is very different, because now the gravitational and degeneracy energies have the same 1/R dependence.

If we now look for a solution to the stability criterion, we find that when the star reaches a critical mass, the 'Chandrasekhar mass, there is a solution to the stability criterion – one has

$$M_{Ch} \equiv \left(\frac{C_2}{C_g G}\right)^{3/2}$$



More refined calculations give the results shown at right