BOSE GASES/LIQUIDS & SUPERFLUIDITY

Recall that to find the chemical potential for 3-d bosons, we wrote

$$N \approx V \int_0^\infty \frac{g(E)}{\exp[\beta(E-\mu)]-1} dE \qquad \text{where} \qquad g(E) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}$$

which gave $$N = \frac{V}{4\pi^2}\left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{x^{1/2}}{\exp[x-\beta\mu]-1} dx$$

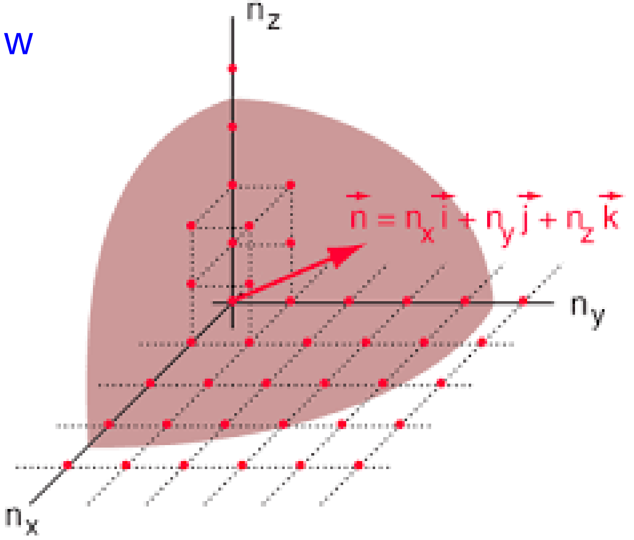
But this integral has a maximum value ~ 1.36 $\pi^{1/2}$, when $\mu = 0$, so that N cannot exceed a maximum value $N = N_{cr}$, given by

$$N_{cr} = \frac{V}{4\pi^2}\left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} 1.36\pi^{1/2}$$

implying a critical density $\rho_{cr} = 2.612\rho_q$

where again $\rho_q = \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2}$

As pointed out previously, this result is wrong. Lets now fix it. The problem is that if $\mu = 0$, we must go back to the discrete density of states if we are to deal with zero energy singularity in the integral above.



The discretized energies are $E_{nlm} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (n^2 + l^2 + m^2)$

so the energy gap between ground state & 1st excited state is

$$E_{211} - E_{111} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (4+1+1-3) = \Delta$$

DISCRETIZING the INTEGRAL

We have, for the Bose function at E = 0, the ground state behaviour

$$f(0) = \frac{1}{\exp[-\beta\mu]-1} \xrightarrow[\mu \to 0]{} \frac{1}{-\beta\mu} = \frac{k_B T}{|\mu|}$$

(diverges as μ → 0, for any T)

On the other hand for the 1st excited state, we have, for $|\mu| << \Delta$ that

$$f(\Delta) = \frac{1}{\exp[\beta(\Delta-\mu)]-1} \sim \frac{1}{\beta\Delta - \beta\mu} \sim \frac{k_B T}{\Delta}$$

(finite as μ → 0, at low T)

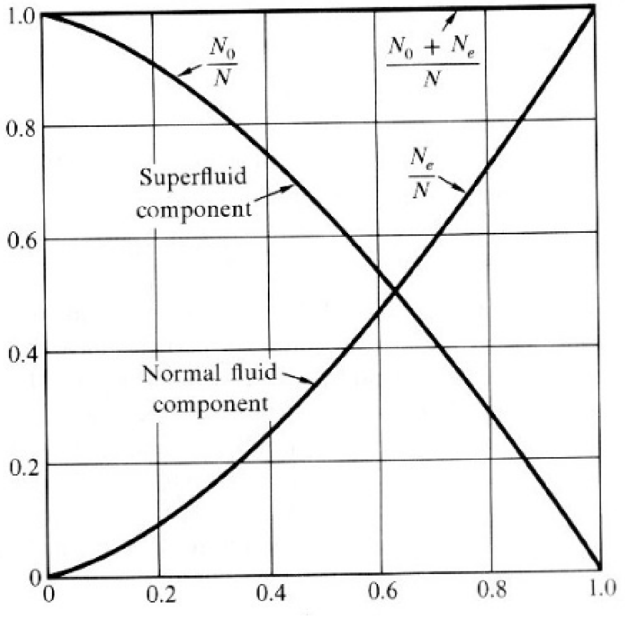
So, in the sum over states for the particle number, let's separate out the ground state; we write:

$$N = \sum_i f(E_i) = f(0) + \sum_{i \neq 0} \frac{1}{\exp[\beta(E_i-\mu)]-1} = \frac{1}{\exp[-\beta\mu]-1} + \frac{V}{4\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{x^{1/2}}{\exp[x-\beta\mu]-1}dx$$

$$\sim \frac{1}{\exp[-\beta\mu]-1} + 2.612V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$$

N_0

excited states



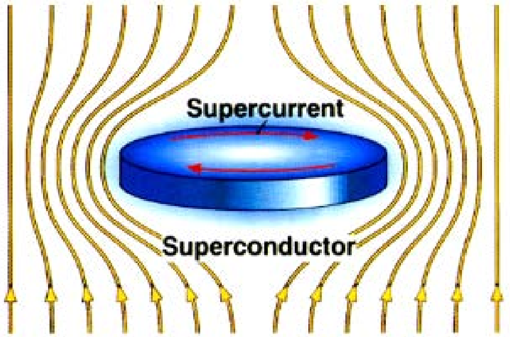
Now how do we deal with the formally divergent term N_0 (which can't diverge if N is fixed!). Let's write

$$\frac{N_0}{N} = 1 - \frac{2.612V}{N}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} = 1 - \left(\frac{T}{T_{BE}}\right)^{3/2}$$

where $T_{BE} = \frac{2\pi\hbar^2}{mk_B}\left(\frac{\rho}{2.612}\right)^{2/3}$ (Bose-Einstein temperature)

Plotting this gives the "normal fluid" (excited state) contribution shown, as a function of $x = T/T_{BE}$

SUPERFLUIDITY

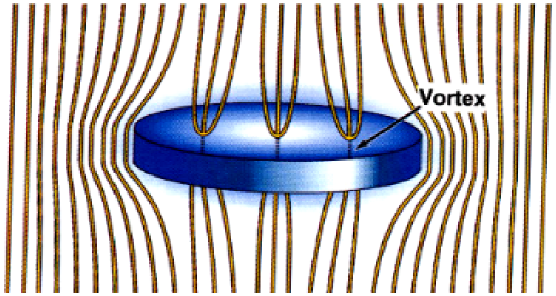


Meissner flux expulsion effect

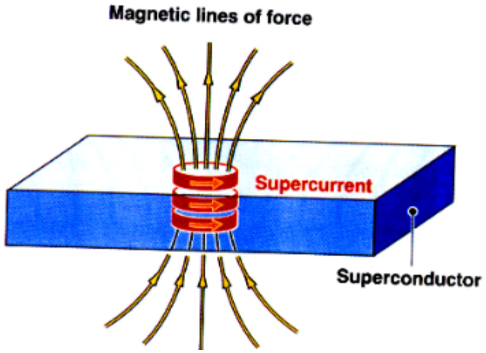
In fermion systems, attractive interactions can cause 2 fermions to “pair” to form a boson (a “Cooper pair”). These can Bose condense to give a SUPERCONDUCTOR.

Superconducting levitation

The tell-tale sign of superconductivity is the Meissner effect – one cools the system in a magnetic field and it expels magnetic flux.



In many superconductors (Type II superconductors), the flux can penetrate via vortex lines, whose flux is quantized. These also show the levitation effect because the flux is pinned by impurities.



For a neutral superfluid like liquid ^{4}He there is a dramatic effect. If one spins a hollow torus packed with liquid ^{4}He and jeweller’s rouge, and then cools into the superfluid, the fluid spontaneously stops rotating! One also gets vortex lines.

The vortex quantization obeys

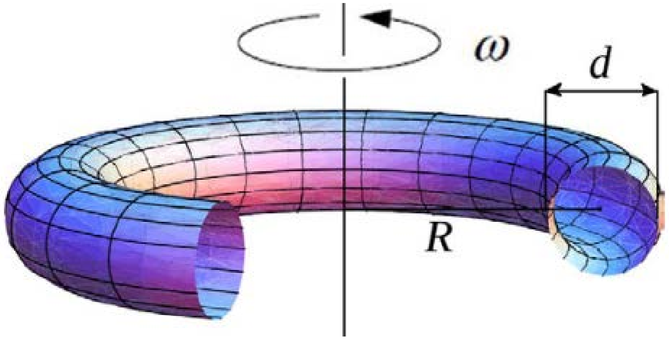
$$\text{Flux} = n\Phi_0$$ Superconductor

$$\text{Circulation} = n\kappa_0$$ Superfluid

where

$$\Phi_0 = h/2e$$ (flux quantum)

$$\kappa_0 = h/m$$ (fluid circulation quantum)



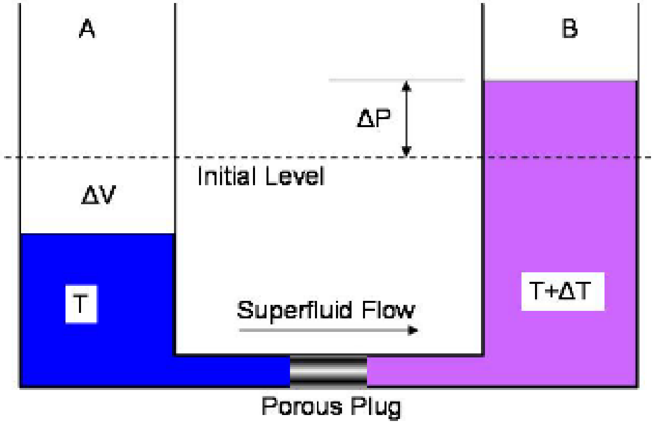
NB: These are equilibrium properties of these systems

SUPERFLOW in NEUTRAL SUPERFLUIDS

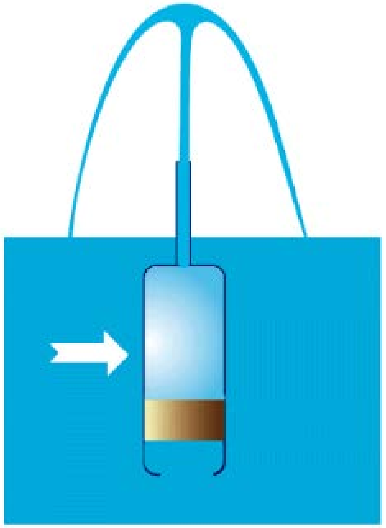
The most dramatic experimental feature of superfluidity is dissipationless superflow - the fluid flows without viscosity through microscopic channels or cracks, up walls, etc., sometimes at high velocity.

EXAMPLES of SUPERFLOW: The first two to be discovered (in 1937) were capillary flow (Kapitza) and the fountain effect (Allen and Misener). Both of these involve high-velocity frictionless flow of superfluid through plugs made from clay or from jeweller's rouge; the cracks or holes are sub-micron in size.

The capillary flow effect is shown here when one has a small temperature difference between superfluid on either side of a plug. Remarkably, superfluid flows "uphill' against the gravitational field.



Experimental fountain effect



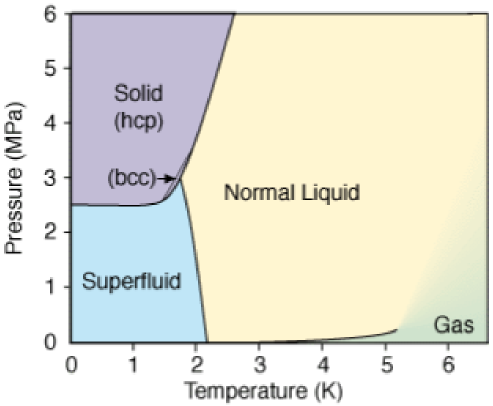
The fountain effect (schematic)

In the fountain effect, a bottle with its bottom cut off has a similar plug. The temperature inside the bottle is slightly raised by shining a light into it. This causes extremely fast flow through the plug, creating a fountain of superfluid rushing out of the top of the bottle.

Superflow in superconductors is called superconductivity – the currents can be v. large, creating huge magnetic fields (used in, eg., MRI machines in hospitals)

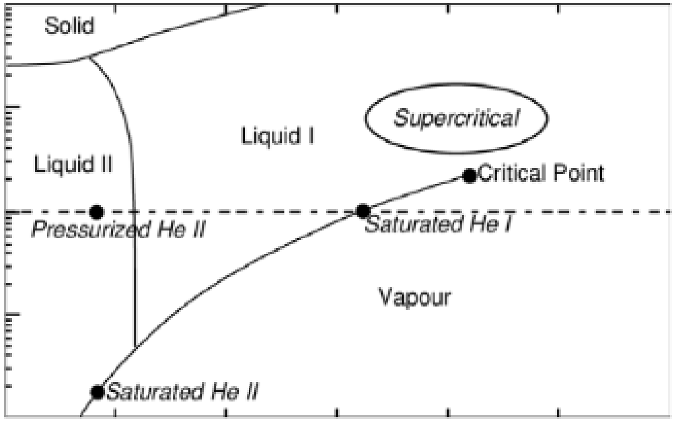
PROPERTIES of SUPERFLUID ^{4}He

PHASE DIAGRAM: A v important part of thermodynamics & stat mech (which we won’t have time to look at) deals with different thermodynamic phases & the transitions between them.

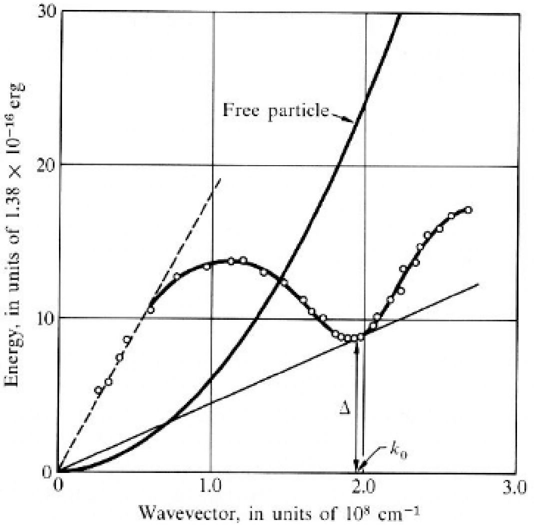


Phase diagram of ^{4}He (P vs T)

In ^{4}He there are four low-T phases. Note the critical point between the gas & liquid states, and the transition of the liquid to superfluid at T ~ 2.2K. There are actually several different solid phases when the pressure p > 25 bar.



Same diagram, but now plotting log P vs T)



Dispersion relation for 4He quasiparticles

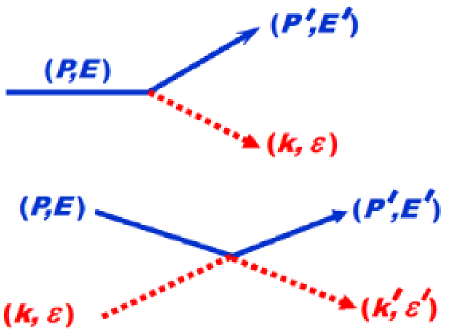
QUASIPARTICLE EXCITATION SPECTRUM: In the normal liquid, friction occurs by scattering of individual particles, with energies $E_p \sim p^2/2m$. However in the superfluid, the low-E excitations are “quasiparticles”; the difference arises from the interactions between the particles.

At left we see the quasiparticle energy $\boldsymbol{\varepsilon}_p$ for superfluid ^{4}He. At low momentum p we have

$$\boldsymbol{\varepsilon}_p = cp \qquad (c = \text{sound velocity})$$

At higher momentum the spectrum turns over – the excitations at the minimum (with energy gap Δ) are called “rotons”.

EXPLANATION for SUPERFLOW PROPERTIES



An external object (in blue) interacts with quasiparticles (in red)
TOP: quasiparticle creation.   BOTTOM: quasiparticle scattering

Suppose we have some object (particle, wire, etc.) moving through a fluid or gas. What causes friction in its motion?

Consider how energy & momentum can be taken from the object by the fluid. This can be done by creating an excitation in the fluid, or by scattering an existing thermal one.

Then we have to conserve energy & momentum, so we have

$$P' - P = k \qquad E' - E = \varepsilon \qquad \text{(quasiparticle creation process)}$$

$$P + k = P' + k' \qquad E + \varepsilon = E' + \varepsilon' \qquad \text{(scattering process)}$$

So let’s send in a mass M with velocity V and see what happens. We have, from above, that

$$M\vec{V} = M\vec{V}' + \hbar\vec{k} \quad \text{(momentum)} \qquad \text{and} \qquad \frac{1}{2}MV^2 = \frac{1}{2}MV'^2 + \varepsilon_{\vec{k}} \quad \text{(energy)}$$

from which we get $\frac{1}{2}MV^2 - \hbar\vec{V}\cdot\vec{k} + \frac{\hbar^2k^2}{2M} = \frac{1}{2}MV'^2$ where $M\vec{V} - \hbar\vec{k} = M\vec{V}'$

Comparing with the energy conservation eqtn, we then get the condition

$$V = \frac{\varepsilon_{\vec{k}} + \frac{\hbar^2k^2}{2M}}{\hbar k} \sim \frac{\varepsilon_{\vec{k}}}{\hbar k}$$

Recoil term

This gives us a graphical condition; dropping the recoil term, we plot the straight line condition $\varepsilon_p = pV$, which is only satisfied for $V > V_c$ (giving a critical velocity)

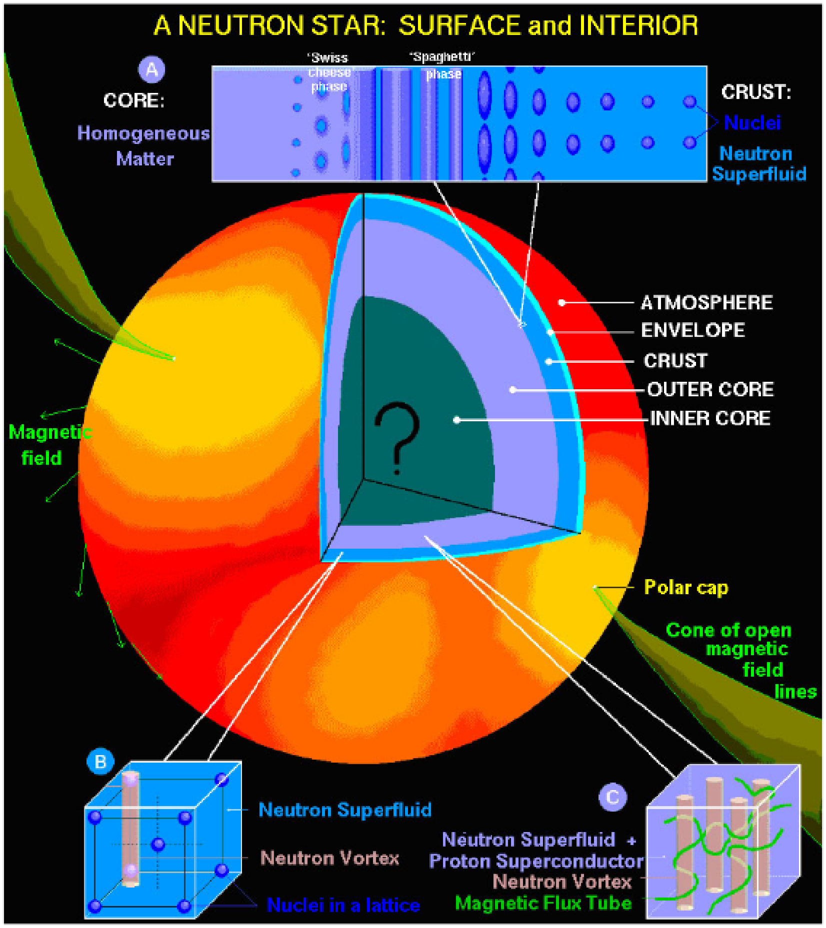
In superfluid ^{4}He, $V_c = \frac{\varepsilon_{\vec{k}}}{\hbar k} = \frac{\Delta}{\hbar k_0} \approx 5\times10^3\, cm\cdot s^{-1}$

NB: one can also excite quantum vortices in the superfluid – this changes the criterion.

NEUTRON STAR SUPERFLUIDS

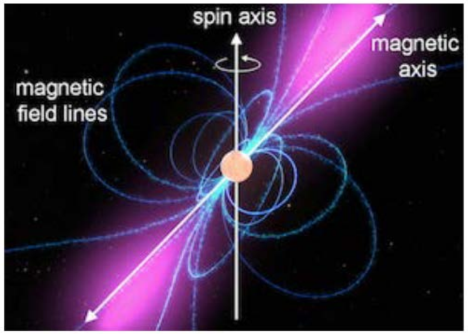
Neutron stars are born in the supernova explosions of supergiant stars. They were first found as "pulsars" in 1967 (Bell & Hewish), but had been predicted theoretically by Landau in 1932.

The surface density is ~ 10^4 g/cc, rising to nearly 10^{15} g/cc in the central core.

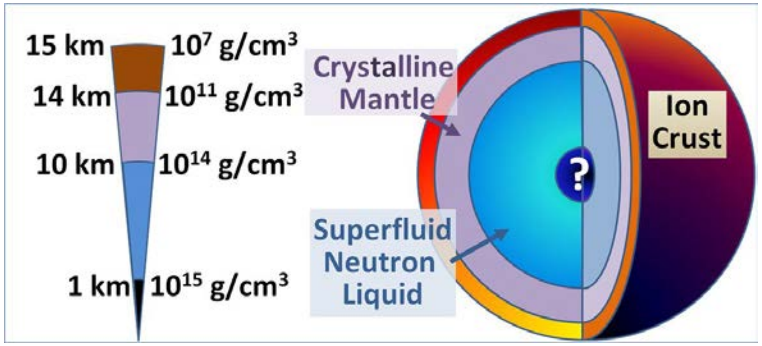


The "Crab Nebula" in Taurus, remnant of a supernova seen in 1054 AD.

The surface is made from squashed spindle-shaped atoms – the core is a mixture of neutron and proton superfluid, permeated by a dense array of vortex lines because the stars rotate really fast. Some of the superfluid is charged, and so the resulting charged supercurrent generates huge magnetic fields (up to ~ 10^{11} Tesla). These huge fields cause processes that make the pulsars visible.



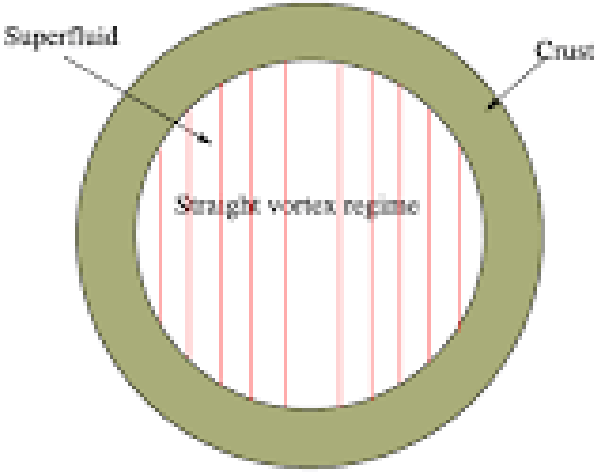
SOME FACTS about NEUTRON STARS



Some of the vital statistics are shown at right – the radii of neutron stars decreases down to less than 7-8 km as their mass Increases. The masses range from ~1.15 M_S up to ~2.16 M_S; above this they are unstable to gravitational collapse unless they are rotating fast.

Rotation rates can be > 2π **x** 700 rads/sec for newly-formed neutron stars, falling by factors of > 100 as they get older (the Crab nebula pulsar, now 1000 yrs old, has a period of 33 msecs). The surface temperature ~ 10^5 K, the core ~ 10^6 K, except immediately after formation when it is ~10^9 K.

The interior is a mixture of neutron and proton superfluids. The Bose condensation temperature is extremely high, ~ 2-3 **x** 10^{10} K. Thus the superfluid is in the extreme low-T regime!



The fast rotation means a high density of vortex lines, spaced in the outer crust roughly 10^{-11} m apart.

To slow its rotation the vortex lattice array needs to ‘depin’ from the crust – this causes observable ‘glitches’ or starquakes, in which the rotation rate suddenly changes.