Physics 403 Problem Set 3

March 23, 2021

Problem 1. Early Universe

At the 'recombination time' τ_R (roughly 400,000 years after the Big Bang), the main constituents of the universe were photons, hydrogen atoms, protons, and electrons. For this problem, we sall ignore the photons and assume that the three remaining species have chemical potentials μ_H, μ_p, and μ_e and number densities n_H, n_p, and n_e respectively. Assume a Hydgron ionization energy E_0. Furthermore, assume that there are two relevant states for the proton and electron (being that they are spin-1/2) and therefore four states for the hydrogen atom.

(a) *Suppose that we can treat this system as low density. What are n_H, n_p, and n_e in terms of μ_H, μ_p, and μ_e?*

Solution: Because the gases are at low density, we can consider them to be non-interacting particles, and treat the ionization and formation of hydrogen to be the particles entering and leaving the system. We also assume that on a sufficiently large scale, the universe is homogeneous. Thus, we can consider some fixed volume V, and let n_a be the number density of the species a in that volume.

We recognize that we can analyze the system using the grand canonical ensemble. Furthermore, because the particles are non-interacting, the partition function Z_G is given by:

$$Z_G = \sum_{N_H} \sum_{N_e} \sum_{N_p} \left(z_G^{(H)}\right)^{N_H} \left(z_G^{(e)}\right)^{N_e} \left(z_G^{(p)}\right)^{N_p}$$

where $z_G^{(a)}$ is the single particle partition function for species a. Notice that the single particle partition function is:

$$z_G^{(a)} = \sum_{\text{states } s} e^{-\beta(E_s - \mu_a)} = z_C^{(a)} e^{\beta \mu_a}$$

for μ_a being the chemical potential of species and a and where z_C is the canonical single particle partition function for species (a). Because we are neglecting any electromagnetic interactions, the only energy contribution to the electron and proton energy will be the kinetic energy. As such, each distinct spin state is described by the free particle partition function in three dimensions, which we know (from Equation 6.26 in the course notes):

$$z_G^{(p)} = 2V \left(\frac{m_p}{2\pi\beta\hbar^2}\right)^{3/2} \qquad z_G^{(e)} = 2V \left(\frac{m_e}{2\pi\beta\hbar^2}\right)^{3/2}$$

The hydrogen atom also has no energy from interactions, but each energy level is shifted by the ionization energy $-E_0$. Thus, the single particle partition function will be:

$$z_G^{(H)} = 4V\left(\frac{m_H}{2\pi\beta\hbar^2}\right)^{3/2} e^{\beta E_0}$$

Note: *This shift in the energy can be seen by considering a hydrogen atom at rest. By adding energy E_0 to the atom, the electron and proton can be separated, giving a proton at rest and an electron at rest. The final system has no energy, so by conservation of energy, the hydrogen atom at rest must have energy $-E_0$. This shifts the minimum value of the Maxwell-Boltzmann distribution, and evaluating the resulting integral causes an additional factor of $e^{\beta E_0}$ to appear in the partition function. Alternatively, we could neglect the energy shift E_0 in the single particle partition function and treat the ionization energy as a correction to the chemical potential. In this case, the ionization energy will not appear in this expression, and will instead appear in the problem in the equilibrium expression in part b.*

Substituting our expressions for the single particle partition functions into our equation for Z_G, we find that:

$$Z_G = \sum_{N_H, N_e, N_p} 2^{N_p+N_e+2N_H} V^{N_p+N_e+N_H} \left(\frac{m_p}{2\pi\beta\hbar^2}\right)^{3N_p/2} \left(\frac{m_e}{2\pi\beta\hbar^2}\right)^{3N_e/2} \left(\frac{m_H}{2\pi\beta\hbar^2}\right)^{3N_H/2} e^{\beta N_H(\mu_H+E_0)} e^{\beta N_e\mu_e} e^{\beta N_p\mu_p}$$

We recognize that this series is the product of geometric series in powers of N_H, N_e, and N_p. Then, using the geometric series formula, we can write:

$$Z_G = \left(\frac{1}{1-2V\rho_H e^{\beta(E_0+\mu_h)}}\right)\left(\frac{1}{1-2V\rho_e e^{\beta\mu_e}}\right)\left(\frac{1}{1-4V\rho_p e^{\beta\mu_p}}\right)$$

where we have defined ρ_a for the species a to be:

$$\rho_a = \left(\frac{m_a}{2\pi\beta\hbar^2}\right)^{3/2}$$

Next, we know that the particle density n_a for species a is given by:

$$n_a = \frac{1}{\beta V Z_G}\frac{\partial Z_G}{\partial \mu_a}$$

Using our expression for Z_G in conjunction with our expression for n_a, we find:

$$n_p = \frac{2\rho_p e^{\beta\mu_p}}{1-2V\rho_p e^{\beta\mu_p}} \qquad n_e = \frac{2\rho_e e^{\beta\mu_e}}{1-2V\rho_e e^{\beta\mu_e}} \qquad n_H = \frac{4\rho_H e^{\beta(\mu_H+E_0)}}{1-4V\rho_H e^{\beta(\mu_H+E_0)}}$$

For each of the species, the quantity $V\rho_a e^{\beta\mu_a}$ is much smaller than 1, so each denominator in the above expressions is approximately equal to 1. Then, rewriting each of the ρ_a in terms of fundamental constants, we have the expressions:

$$\boxed{n_p = 2\left(\frac{m_p}{2\pi\beta\hbar^2}\right)^{3/2} e^{\beta\mu_p}} \qquad \boxed{n_e = 2\left(\frac{m_e}{2\pi\beta\hbar^2}\right)^{3/2} e^{\beta\mu_e}} \qquad \boxed{n_H = 4\left(\frac{m_H}{2\pi\beta\hbar^2}\right)^{3/2} e^{\beta(\mu_H+E_0)}}$$

(b) *What defines thermal equilibrium for this system? At equilibrium, what are n_H, n_e, and n_p?*

Solution: At equilibrium for the system, the rate at which each species enters the system is the same as the rate at which the species leaves the system, which requires that the chemical potentials for the electron and proton The energy required to remove a hydrogen atom from the system is μ_H. The energies required to remove an electron and a proton from the system are μ_e and μ_p respectively. Thus, we have the constraint:

$$\boxed{\mu_H = \mu_e + \mu_p}$$

Next, using our expressions from the previous part, we see that:

$$\frac{n_e n_p}{n_H} = \left(\frac{m_p m_e}{2\pi\beta\hbar^2 m_H}\right)^{3/2} e^{\beta(\mu_e+\mu_p-\mu_H-E_0)}$$

Using our equilibrium condition, we see that this simplifes to the following result:

$$\boxed{\frac{n_e n_p}{n_H} = \left(\frac{m_p m_e}{2\pi\beta\hbar^2 m_H}\right)^{3/2} e^{-\beta E_0}}$$

(c) *Using values for E_0 and for the mass m_e of an electron that can be found in the literature, find the density n_e when $n_H = n_p = n_e$ (that is, when half of the hydrogen atoms are ionized). This is the density at the recombination time τ_R.*

Solution: When $n_e = n_p = n_H$, our expression from the previous part becomes:

$$n_e = \left(\frac{m_p m_e}{2\pi\beta\hbar^2 m_H}\right)^{3/2} e^{-\beta E_0} = \left(\frac{m_p m_e}{2\pi\beta\hbar^2 (m_p + m_e - E_0)}\right)^{3/2} e^{-\beta E_0}$$

Expanding the denominator to leading order about $m_e - E_0 \approx 0$, we find:

$$n_e \approx \left(\frac{m_p m_e}{2\pi\beta\hbar^2 m_p}(1 + E_0 - m_e)\right)^{3/2} e^{-\beta E_0} \approx \left(\frac{m_e}{2\pi\beta\hbar^2}\right)^{3/2} e^{-\beta E_0}$$

where we have dropped all terms that are higher than first order in products of m_e and E_0. The ground state ionization energy for the hydrogen atom (which is a very useful number to have memorized) is $E_0 = 13.6$ eV, the mass of the electron is 0.511 MeV/c^2, and the the recombination temperature is 3740 K. Substituting these values into our previous expression, we find:

$$n_e = \left(\frac{(0.511 \times 10^6 \text{ eV})(8.62 \times 10^{-5} \text{ eV/K})(3740 \text{ K})}{2\pi(6.58 \times 10^{-16} \text{ eV}\cdot\text{s})^2(3.0 \times 10^8 \text{ m/s})^2}\right)^{3/2} e^{-(13.6 \text{ eV})/(8.62\times10^{-5} \text{ eV/K})(3740 \text{ K})}$$

Evaluating this result, we find that the electron number density is:

$$\boxed{n_e = 2.64 \times 10^8 \text{ m}^{-3}}$$

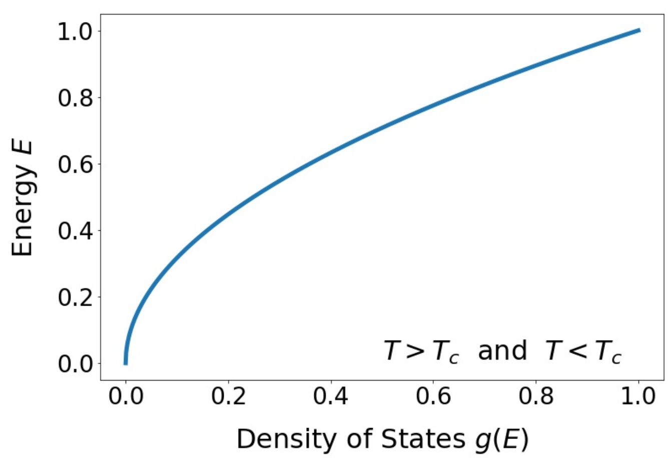
Problem 2. Bose Gases

(a) *Draw two graphs as a function of the energy E which shows the one-particle density of states, and the Bose distribution function for a three-dimensional Bose-System of massive particles with mass m for the cases $T > T_c$ and $T < T_c$, where T_c is the BEC condensation temperature. Then, draw two graphs showing the product of these two functions, as a function of energy in each of the two temperature cases.*

Solution: We consider the Bose gas as a homogeneous gas and restrict our attention to a box with unit volume. We know (from Equation 4.58) that the single particle density of states $g(E)$ is:

$$g(E) = \frac{\sqrt{E}}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$$

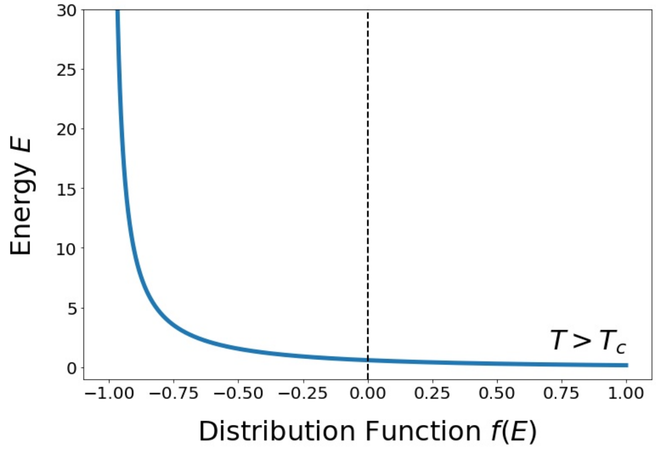
Notice that this expression is neither explicitly nor implictly (through the chemical potential) dependent on the temperature of the ensemble, so the density of states above and below the critical temperature will be the same. A plot of the degeneracy of states with dimensions chosen such that $g(E) = \sqrt{E}$ is shown below:

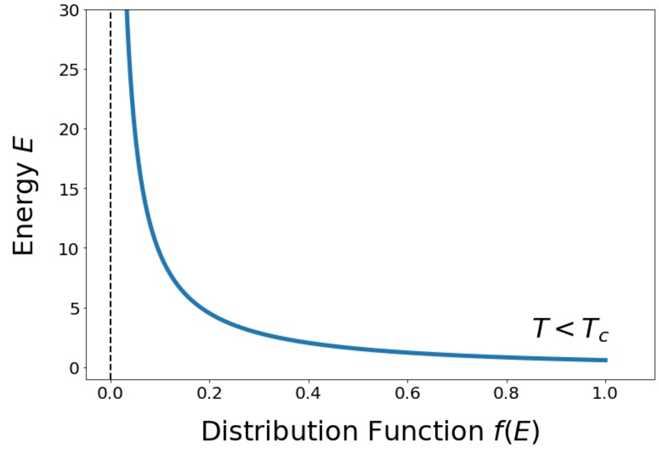


Next, a Boson gas obeys the Bose distribution $f(E)$, which we know (from Equation 7.18) is given by

$$f(E) = \frac{1}{e^{\beta(E-\mu)} - 1}$$

where μ is the chemical potential of the gas. For $T > T_c$, the chemical potential is negative, and the distribution has a finite value at $E = 0$. For $T < T_c$, the chemical potential vanishes and the distribution diverges at $E = 0$. The plots of these two cases with dimensions chosen such that $\beta = 1$ and such that $\mu = 1$ for $T > T_c$ are shown below:

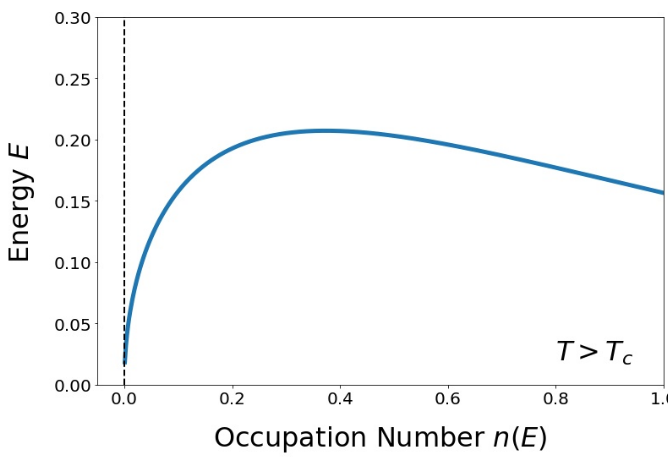


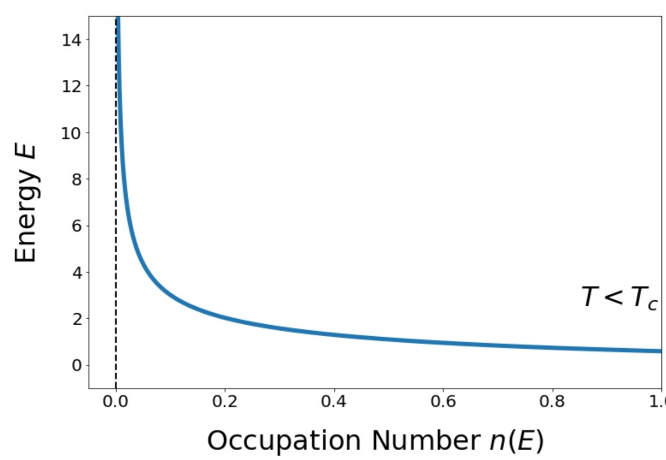


Finally, the number of particles $n(E)$ with an energy E is given by the product of the density of states and the Bose distribution. Thus:

$$n(E) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\sqrt{E}}{e^{\beta(E-\mu)}-1}$$

Once more, for $T > T_c$, the chemical potential is negative. For $T < T_c$, the chemical potential vanishes. Below, we see plots of n for T above and below T_c with the dimensional scalings used in the previous plots:





(b) *A criterion for a BEC to occur in a three dimensional gas of bosons is that the chemical potential* $\mu = 0$. *Explain this criterion with reference to the relevant mathematical expressions.*

Solution: Physically, we can understand the chemical potential μ as being the negative of the energetic cost required to add a particle to the system. Thus, when $\mu < 0$, it is energetically unfavorable to add particles to the system, and we cannot add infinitely many particles to the system. However, when we have $\mu = 0$, there is no energetic cost to add particles to the zero-energy ground state and only to this ground state. Thus, some arbitrarily large number of particles must be in the ground state, leading to a condensate. We shall now mathematically verify this physical intuition.

The total number of particles N in the system is given by integrating the number of particles with an energy E over all energies along with the number of particles N_0 in the ground state. That is:

$$N = N_0 + \int_0^\infty dE\; n(E)$$

The ground state is the zero energy state with zero momentum, which has degeneracy 1. Thus, N_0 is just the value of the Bose distribution at $E = 0$. Using this along with our expression for $n(E)$ from the previous part, we find:

$$N = \frac{1}{e^{-\beta\mu} - 1} + \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty dE\; \frac{\sqrt{E}}{e^{\beta(E-\mu)} - 1}$$

Using the geometric series formula, we can write the denominator of the integrand as a series in the following way:

$$N = \frac{1}{e^{-\beta\mu} - 1} + \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sum_{k=1}^{\infty} \int_0^\infty dE\; \sqrt{E} e^{k\beta(\mu - E)}$$

We can now use the substitution $u = k\beta E$ to rewrite this integral as:

$$N = \frac{1}{e^{-\beta\mu} - 1} + \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sum_{k=1}^{\infty} \frac{e^{k\beta\mu}}{(k\beta)^{3/2}} \int_0^\infty du \sqrt{u} e^{-u}$$

We recognize that the integral is a definition for the Γ-function evaluated at $3/2$, which is equal to $\sqrt{\pi}/2$. We also recognize that the series is the definition of the polylog function $\mathrm{Li}_{\frac{3}{2}}$. Then, we can rewrite this expression as:

$$N = \frac{1}{e^{-\beta\mu} - 1} + \frac{1}{2}\left(\frac{m}{\hbar^2 \beta\pi}\right)^{3/2} \mathrm{Li}_{\frac{3}{2}}(e^{\beta\mu})$$

We know that the chemical potential μ cannot be positive, so $e^{\beta\mu}$ is restricted to the interval $(0, 1]$. The polylogarithim function is bounded on this interval, so only finitely many particles can be added to the excited states of the system at any finite temperatures. Thus, all particles added to the system above this bound must be added to the ground state, and we have the constraint that for a BEC to form:

$$N_0 = \frac{1}{e^{-\beta\mu} - 1}$$

must diverge. This occurs precisely when $\mu = 0$, verifying our constraint. $\square$

(c) *Rederive the criterion for two-dimensional and one-dimensional systems. What do the results tell you about BECs in these cases?*

Solution: We know (from Equation 4.54) that the density of states $g_1(E)$ in one dimension is given by:

$$g_1(E) = \frac{1}{\pi\hbar}\sqrt{\frac{m}{2E}}$$

Multiplying by the Bose-Einstein distribution gives us $n_1(E)$, the number of particles at an energy E is:

$$n_1(E) = \frac{1}{\pi\hbar}\sqrt{\frac{m}{2E}} \frac{1}{e^{\beta(E-\mu) - 1}}$$

The total number of particles N_1 is given by the integral of n_1 over all N along with the number of particles N_0 in the ground state. We know that N_0 is just the Bose distribution at $E = 0$. Then:

$$N_1 = \frac{1}{e^{-\beta\mu} - 1} + \frac{1}{\pi\hbar}\sqrt{\frac{m}{2}} \int_0^\infty dE \; \frac{1}{\sqrt{E}} \frac{1}{e^{\beta(E-\mu)} - 1}$$

As we did in the previous part, we write the integrand as an infinite geometric series then use the substitution $u = \beta k E$ to rewrite the integral as:

$$N_1 = \frac{1}{e^{-\beta\mu} - 1} + \frac{1}{\pi\hbar}\sqrt{\frac{m}{2\beta}} \sum_{k=1}^{\infty} \frac{e^{k\beta u}}{\sqrt{k}} \int_0^\infty du \; u^{-1/2} e^{-u}$$

The integral is the definition of the Γ-function evaluated at $1/2$, which is $\sqrt{\pi}$. Furthermore, we see that the series is the definition of the polylog function $\mathrm{Li}_{\frac{1}{2}}$. Then, we can rewrite the previous expression as:

$$N_1 = \frac{1}{e^{-\beta\mu} - 1} + \sqrt{\frac{m\pi}{2\beta\hbar^2}} \mathrm{Li}_{\frac{1}{2}}(e^{\beta\mu})$$

In order to have N_1 diverge, one of these terms must diverge. We have seen in the previous parta that the first term diverges precisely as μ approaches 0. The polylog function of order $1/2$ diverges to infinity when approaching from below 1, which tells us that $\mathrm{Li}_{\frac{1}{2}}(e^{\beta\mu})$ diverges to infinity as μ approaches 0 from below. Thus, our contraint for N_1 to be arbitrarily large is that $\mu = 0$. Then, we see that both the ground state occupation and excited state occupations diverge together, and there is therefore no BEC.

We shall now repeat the calculation in two dimensions. We know (from Equation 4.57) that the density of states $g_2(E)$ in two dimensions is given by:

$$g_2(E) = \frac{m}{2\pi\hbar^2}$$

Following the procedure we performed in one dimension, the number of particles N_2 in the system is given by:

$$N_2 = \frac{1}{e^{-\beta\mu} - 1} + \frac{m}{2\pi\hbar^2} \int_0^\infty \frac{dE}{e^{\beta(E-\mu)} - 1} = \frac{1}{e^{-\beta\mu} - 1} + \frac{m}{2\pi\hbar^2\beta} \sum_{k=1}^{\infty} \frac{e^{k\beta\mu}}{k} \int_0^\infty du \; e^{-u}$$

The integral evaluates to 1, and we recognize that the series is the Taylor expansion of the logarithm function. Then, we can write:

$$N_2 = \frac{1}{e^{-\beta\mu} - 1} - \frac{m}{2\pi\hbar^2\beta} \log\left(1 - e^{\beta\mu}\right)$$

Once again, both terms diverge precisely at $\mu = 0$. However, because both the ground state occupation and excited state occupation diverge together, there cannot be a BEC. $\square$

(d) *Consider a photon gas. Why is $\mu = 0$ always true for photons? Derive an expression for the energy density $U(T)$ for a photon gas in n-dimensions for n being some positive integer. Show that $U(T) \propto T^{n+1}$.*

For a system of non-interacting massless bosons, such as a photon gas, the energy of the system has no dependence on the number of particles, only on the sums of the momenta of the particles. For example, a system of two photons with momenta $\hbar k_1$ and $\hbar k_2$ respectively has the same energy as a single photon with momentum $\hbar(k_1 + k_2)$. There is subsequently no energy cost to adding particles into the system, so we must have the chemical potential $\mu = 0$. Then, the Bose distribution f_γ for photons becomes:

$$f_\gamma(E) = \frac{1}{e^{\beta E} - 1}$$

We shall now determine the density of states g_γ for a gas in n-dimensions. We consider a gas constrained to a n-dimensional box with side length L. We have the condition that $\vec{k}$ must vanish at the boundary of the box. Then, each component k_j of the wave-vector must satisfy:

$$k_j = \frac{m_j \pi}{L}$$

for positive integers m_j. This tells us that each allowed wave-vector component must be separated by a minimum of π/L, so each allowed wave-vector occupies an n-dimensional volume $(\pi/L)^n$ in k-space. The number of states G with energy less than E is given by:

$$G(E) = \frac{1}{2^n} \int_0^{k(E)} d^n\vec{k}$$

where we divide by 2^n to account for the fact that the wave-vector modes are equivalent under sign changes in the components, so we only need to consider the modes where all components are positive. Writing this in n-spherical coordinates, we have:

$$G(E) = \frac{1}{2^n} \int_0^{k(E)} dk \; d\Omega_{n-1} \; k^{n-1}$$

where $d\Omega_{n-1}$ is the solid angle of the $n-1$-sphere. The integrand is spherically symmetric, so the integral over the solid angle yields the volume of the unit $n-1$ sphere. Then, we can write:

$$G(E) = \frac{2\pi^{n/2}}{2^n \Gamma(n/2)} \int_0^{k(E)} dk \; k^{n-1}$$

> ***Note:*** *The unit $n-1$ sphere is an $n-1$-dimensional surface, so its volume is the surface area of the unit $n-1$ ball. Checking the prefactor above for $n = 2$ and $n = 3$ yields the familiar results of 2π (the 1-dimensional "volume" of the circumference of the unit circle) and 4π (the 2-dimensional "volume" of the surface area of a sphere).*

Next, we know that the energy E of a photon with wave-vector $\vec{k}$ is given by the relativistic equation:

$$E = \hbar c k$$

Then, we can rewrite our integral expression for $G(E)$ as:

$$G(E) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^{E} dE' \; \frac{(E')^{n-1}}{(2\hbar c)^n}$$

The degeneracy of states $g_\gamma(E)$ is defined to be the number of states in the system between E and $E+dE$. We recognize that this is the infinitesimal increase dG over an increase dE, or:

$$g_\gamma(E) = \frac{dG}{dE} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{E^{n-1}}{(2\hbar c)^n}$$

The energy density U of the system is given by integrating the product of the energy and the number of particles at that energy level over all energies. The number of particles at a given energy level is the product of the degeneracy af states g_γ with the distribution f_γ. Then:

$$U = \frac{2\pi^{n/2}}{\Gamma(n/2)(2\hbar c)^n} \int_0^\infty dE \; \frac{E^n}{e^{\beta E} - 1}$$

As we did in the previous parts, we can write the denominator of the integrand as a geometric series and use the substitution $u = \beta E$, which yields:

$$U = \frac{2\pi^{n/2}}{\Gamma(n/2)(2\hbar c)^n} \sum_{k=1}^{\infty} \frac{1}{k\beta^{n+1}} \int_0^\infty du \; u^n e^{-u}$$

Once more writing the integral as a Γ-function and the series as a polylog Li_{n+1}, we have:

$$U = \frac{2\pi^{n/2}\Gamma(n+1)\mathrm{Li}_{n+1}(1)}{\Gamma(n/2)(2\hbar c)^n \beta^{n+1}}$$

Rewriting this expression in terms of the temperature T, we find:

$$\boxed{U = \frac{2\pi^{n/2}\Gamma(n+1)\mathrm{Li}_{n+1}(1)(k_B T)^{n+1}}{\Gamma(n/2)(2\hbar c)^n}}$$

Problem 3. Superfluids

(a) *Consider mass m moving through a fluid with constant viscousity coefficient η. Find the equation of motion of the particle, assuming there is an external force $f(t)$ acting on it. Suppose that the initial velocity at the time $t = 0$ satisfies $v(0) = v_0$. Show that the solution to this equation of motion is:*

$$v(t) = v_0 e^{-\gamma t} + \int_0^t ds \; \frac{f(s)}{m} e^{\gamma(s-t)}$$

where $\gamma = \eta/m$. Then, show that if $f(t)$ goes to a constant f_0 at long times, then the particle will reach a constant velocity v_f. What is v_f?

There are two forces on the particle, the external force $f(t)$ and the drag force $-\eta v$. Then, Newton's Equation of Motion tells us that:

$$M\frac{dv}{dt} = f(t) - \eta v(t)$$

Rearring this expression and multiplying both sides by $e^{\eta t/m}$, we find that:

$$\frac{dv}{dt}e^{\eta t/m} - \frac{\eta}{m}ve^{\eta t/M} = \frac{f(t)}{m}e^{\eta t/m}$$

Using the product rule, we can rewrite the previous expression that:

$$\frac{d}{dt}\left(ve^{\eta t/m}\right) = \frac{f(t)}{m}e^{\eta t/m}$$

Integrating both sides from 0 to t, we find:

$$v(t)e^{\eta t/m} - v(0) = \int_0^t ds\ \frac{f(s)}{m}e^{\eta s/m}$$

Solving for $v(t)$, defining $\gamma = \eta/m$, and using our initial condition that $v(0) = v_0$, we find:

$$\boxed{v(t) = v_0 e^{-\gamma t} + \int_0^t ds\, \frac{f(s)}{m}e^{\gamma(s-t)}}$$

We can rewrite our expression for v in the following way:

$$v(t) = v_0 e^{-\gamma t} + \frac{e^{-\gamma t}}{m\gamma}\int_0^t ds\ f(s)\frac{d}{ds}e^{\gamma s}$$

Now, we can integrate by parts to find:

$$v(t) = v_0 e^{-\gamma t} + \frac{f(t)}{m\gamma} - \frac{f(0)e^{-\gamma t}}{m\gamma} - \frac{e^{-\gamma t}}{m\gamma}\int_0^t ds\ f'(s)e^{\gamma s}$$

Taking the limit as t goes to infinity, we find:

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\frac{f(t)}{m\gamma} - \lim_{t\to\infty}\frac{e^{-\gamma t}}{m\gamma}\int_0^t ds\ f'(s)e^{\gamma s}$$

We define the final velocity v_f to be the limit of the (magnitude of the) velocity in large times. We know that $f(t)$ converges to f_0 in long times, so the first limit is a constant. This also tells us that $f'(t)$ must vanish in long times. Then, for sufficiently large times, the integrand becomes arbitrarily small, and the limit will become dominated by the exponentially decaying prefactor. Thus, the expression vanishes, and we are left with:

$$\boxed{v_f = \frac{|f_0|}{m\gamma}}$$

Note: *This argument holds for almost any function f that we would encounter in a physical system, but it does not hold in general, namely for functions f that are not uniformly continuous close to infinity. For those that are not satisfied, an argument can also be made by establishing limits on how far $|f(t)|$ deviates from $|f_0|$ in some region of large t, bounding the integrand and evaluating the resulting integral, and showing that the result converges in the limit where t is large.*

(b) *In a superfluid, the friction depends on the velocity. Suppose that* $\eta(v) = \eta_0(v - v_c)\Theta(v - v_c)$ *where* η_0 *is a constant and where* $\Theta(x) = 0$ *for* $x < 0$ *and* $\Theta(x) = 1$ *for* $x > 0$*. Find the new terminal velocity* v_f *without solving the new equations of motion.*

Solution: When the particle is at terminal velocity, it will not be accelerating. Then, the equation of motion found in the previous part becomes:

$$0 = \lim_{t\to\infty} \Big(f(t) - \eta(v)v(t) \Big)$$

Substituting in our expression for η, we find:

$$0 = \lim_{t\to\infty} \Big(f(t) - \eta_0(v - v_c)\Theta(v - v_c)v(t) \Big)$$

Evaluating the limits and rearranging the expression, we find:

$$|f_0| = \eta_0(v_f - v_c)v_f\Theta(v_f - v_c)$$

If f_0 vanishes, then the right hand side must vanish as well. This requires that $\Theta(v_f - v_c) = 0$, and we have the constraint that $v_f < v_c$. If not, then we must have $v_f > v_c$, because the right hand side cannot vanish. In the second case, the expression becomes:

$$|f_0| = \eta_0(v_f - v_c)v_f$$

Solving this expression for v_c, we find that:

$$v_f = \frac{v_c}{2} \pm \sqrt{\frac{4|f_0| + \eta_0 v_c^2}{4\eta_0}}$$

Notice that we must have $v_f > v_c$, which requires that we take the positive sign. Then, we conclude that:

$$\boxed{v_f = \frac{v_c}{2} + \sqrt{\frac{4|f_0| + \eta_0 v_c^2}{4\eta_0}}}$$

(c) *Superfluids have quantized vortex ring excitations. For a circular ring of radius* R*, the energy* E *and momentum* p *are of the form:*

$$E \sim \frac{1}{2}\rho\kappa^2 R \log\left(\frac{R}{a_0}\right) \qquad\qquad p \sim \pi\rho\kappa R^2$$

where ρ *is the superfluid density,* κ *is the circulation quantum, and* $a_0 \sim 0.1$ *nm is a vortex core radius. Suppose that the critical velocity* v_c *for the formation of a vortex ring is given by:*

$$v_c \sim \min\left\{\frac{E}{p}\right\}$$

Show that in an infinite system, we have $v_c \to 0$*. Find* v_c *if the superfluid is moving through a cylindrical tube of radius* R_0*. Finally, the vortex ring velocity* v *is given by:*

$$v = \frac{dE}{dp}$$

Find v as a function of the radius R, and sketch a graph of it.

Solution: Using our expressions for v_c, E, and p, we find:

$$v_c \sim \min_p \left\{ \frac{E}{p} \right\} = \min_p \left\{ \frac{\frac{1}{2}\rho\kappa^2 R \log\left(\frac{R}{a_0}\right)}{\pi\rho\kappa R^2} \right\} = \min_p \left\{ \frac{\kappa}{2\pi R} \log\left(\frac{R}{a_0}\right) \right\}$$

Next, using the chain rule, we see that:

$$\frac{d}{dp}\left(\frac{E}{p}\right) = \frac{d}{dR}\left(\frac{E}{p}\right)\frac{dR}{dp} = \frac{d}{dR}\left(\frac{E}{p}\right)\left(\frac{dp}{dR}\right)^{-1} = \frac{1}{4\pi^2\rho R^3}\left\{1 - \log\left(\frac{R}{a_0}\right)\right\}$$

We see that this function is positive for $R < a_0 e$ and is negative for $R > a_0 e$. Furthermore, R must be much larger than a_0 to physically realize a vortex, that is $R \gg a_0$. Thus, in the region of interest, the derivative is strictly negative, so E/p is monotonically as R increases and therefore as p increases. Then, E/p is minimized at the maximum possible value of p, which is at the maximum value of R.

Note: *The vortex core size provides a scale to internal structure of the superfluid. It can be shown that this length is the minimum length over which the wave-function can change. Thus, as the vortex is a region over which the behavior of the superfluid differs from the overall flow of the larger system, it must be much larger than a_0.*

Now, in an infinite system, the radius R of the ring can be arbitrarily large. Furthermore, we see that:

$$\lim_{R\to\infty} \frac{\kappa}{2\pi R} \log\left(\frac{R}{a_0}\right) = 0$$

Thus, the minimum value of E/p goes to 0 as R becomes infinitely large, and:

$$\boxed{\lim_{R\to\infty} v_c = 0}$$

Suppose now that the superfluid is constrained to a tube of radius R_0. Then, v_c is minimized for vortices with radius R_0, and we have:

$$v_c \sim \min\left\{ \frac{\kappa}{2\pi R_0} \log\left(\frac{R_0}{a_0}\right) \right\}$$

The critical velocity is smallest for the lowest circulation quantum, that is when $\kappa = 1$. Thus:

$$\boxed{v_c = \frac{1}{2\pi R_0} \log\left(\frac{R_0}{a_0}\right)}$$

Finally, using the chain rule and the inverse function theorem, we see that:

$$v = \frac{dE}{dp} = \frac{dE}{dR}\frac{dR}{dp} = \frac{dE}{dR}\left(\frac{dp}{dR}\right)^{-1} = \frac{\kappa^2\rho}{2}\left\{1 + \log\left(\frac{R}{a_0}\right)\right\}\left(\frac{1}{2\pi R\rho\kappa}\right)$$

Simplifying this result, we find that velocity is given by:

$$\boxed{v = \frac{\kappa}{4\pi R}\left[1 + \log\left(\frac{R}{a_0}\right)\right]}$$

We have plotted our expression for $v(R)$ below with $\kappa = 1$ and units such that $a_0 = 1$: