PHYS 403: FINAL EXAM 2021 (from 08.30 hrs Apr 27th to 14.30 hrs Apr 28th)

This is a FINAL EXAM. It will last for **30 HOURS**, beginning at 08.30 hrs (8.30 am Vancouver time) on April 27th 2021, and finishing at 14.30 hrs (2.30 pm Vancouver time) on April 28th 2021. You should make **no attempt to collaborate, or discuss this exam, with any other person**.

You must upload your finished exam **BEFORE 14.30 hrs (2.30 pm) on APRIL 28th**, to the canvas webpage. If for some reason this fails, you can call upon either me, using the Zoom link, or upon the TAs who will be helping to supervise the exam. If you still can't upload it, then you should email it directly to me at my email: **stamp@phas.ubc.ca**.

You can either type your exam and then just upload the file, or write it out by hand, and then either scan or photograph it, and upload the file. If you do the latter, then please make sure to write LEGIBLY and NOT IN PENCIL (which scans and photographs very badly); Please use black pen if you can (blue ink also scans and photographs badly). Let me emphasize - what we cannot read we will not mark.

If you have questions during the exam, then the best thing to do is try to contact me or a TA over Zoom during the times we will be available, or by email otherwise (I will be looking constantly at my email). If all else fails, and you really urgently need help, call me at (604) 209-5777. Please do NOT call me unless it is really important! For the Zoom link: At least one of us will try to be available on the Zoom link at all times between 08.30-22.00 hrs on Tuesday, and between 08.00-15.00 hrs on Wednesday (Vancouver time).

EXAM INSTRUCTIONS: The exam is divided into 8 short questions (in section A) and 4 long questions (in section B). You must answer a total of **FOUR** short questions from section A, and **TWO** long questions from section B. You can choose which questions you decide to answer. Note that extra marks will not be given for answering more than 4 questions in section A or 2 questions in section B; if you do, we will simply choose those questions which give you the highest mark.

SECTION A: SHORT QUESTIONS (ANSWER 4 of THESE)

QUESTION A.1: QUANTUM GASES

(i): Why does the diameter of a white dwarf decrease when its mass increases?

(ii) Why does the chemical potential of a gas (Bose, Fermi, or classical) never increase (and almost always decreases) as one raises the temperature?

QUESTION A.2: SPECIFIC HEAT

(i): Consider a set of N non-interacting 2-level systems. What is the difference $\Delta S = [S(T = \infty) - S(T = 0)]$ for this system?

(ii) Suppose we can approximate the specific heat $C_V(T)$ of this system by the simple formula

$$C_V(T) \;=\; C_o \left[1 - 4 \left(\frac{T - T_o}{T_o} \right)^2 \right]$$

for $T_o/2 < T < 3T_o/2$, and zero otherwise. Using the relation between the specific heat and the entropy, and the result you found for ΔS in (i) above, find the value of C_o.

QUESTION A.3: 2-LEVEL SYSTEMS

(i): Consider a set of N non-interacting 2-level systems (TLS), with level energies E_1 and E_2 for each of the TLS. At temperature T, what is the average energy $U(T)$ for the total system? Derive also the specific heat $C_V(T)$.

(ii) Find expressions for $U(T)$ and $C_V(T)$ when $kT \gg |E_1 - E_2|$. You should find the $T = \infty$ result, and also the first correction to this result, for finite (but very large) T.

QUESTION A.4: FERMI DISTRIBUTION

(i): The grand canonical partition function for a single fermion state of energy ϵ is $z(\epsilon) = \sum_n \exp[n\beta(\mu - \epsilon)] = 1 + \exp[\beta(\mu - \epsilon)]$. Show that the mean occupation number for this state is just the Fermi function, ie., that $\langle n \rangle \rightarrow f(\epsilon - \mu) \equiv \{1 + \exp[\beta(\epsilon - \mu)]\}^{-1}$, which we also write as $f(x) = [1 + e^{\beta x}]^{-1}$, where $x = (\epsilon - \mu)$.

(ii) Then show that the probability of finding n particles in this state is

$$p(n) \;=\; \frac{[1 - f(-x)]^n}{[f(-x)]^{n-1}}$$

QUESTION A.5: INTERATOMIC POTENTIAL

(i): Consider the 1-dimensional potential

$$V(x) \;=\; V_o \left[\left(\frac{a_o}{x} \right)^{12} - 2 \left(\frac{a_o}{x} \right)^{6} \right]$$

Find the value of x for which $V(x)$ is a minimum, and find the "curvature" d^2V/dx^2 at this point. What is the frequency of small oscillations of a particle of mass M about the minimum in this potential?

(ii) Draw a picture of the potential $V(x)$, and explain briefly how it can be used to model interatomic interactions. For such interaction, what do you think are typical values for V_o and a_o?

QUESTION A.6: ARGON in ATMOSPHERE

(i): Roughly 1 percent of the volume of the earth's atmosphere is composed of ^{40}Ar. Suppose you are in a bedroom with a volume of 60 m^3. Roughly how many ^{40}Ar atoms are in the room, and what is their total mass?

(ii) In MKS units, roughly what is the total thermal energy associated with the ^{40}Ar atomic motion?

QUESTION A.7: NEGATIVE TEMPERATURE

(i): A set of non-interacting or very weakly interacting spin-1/2 spins has an entropy which looks roughly like $S(U) = S_o - \alpha U^2$, for $U^2 < \alpha$, as a function of the total energy U, and is zero for $U^2 > \alpha$. from the definition of temperature T in terms of S and U for a system in equilibrium, find U in terms of T, and sketch a graph of it.

(ii) What is the specific heat of this system, in the temperature range $-\infty < T < \infty$? How do you interpret this result for $T < 0$?

QUESTION A.8: RADIATION PRESSURE The radiation pressure p from photons is equal to $p = 4J/3c$, where J is the radiation flux. A star like the sun emits black-body radiation with flux $J = \sigma T^4$ per unit area of its surface, where temperature T is measured in Kelvin units; here $\sigma = 5.67 \times 10^{-8} W m^{-2} K^{-4}$, and the sun's surface temperature is $6,000\ K$. The radius of the sun is $R_S \sim 0.7 \times 10^6$ km.

(i): Consider the forces on an electron at the sun's surface. If the cross-section for photon-electron scattering is $\sim 6.6 \times 10^{-29}\ m^2$, and the electron mass is $\sim 9 \times 10^{-31}$ kg, then how do the gravitational and radiation forces on the electron at the sun's surface compare (assume here that all the photon energy is taken up by the electron)? You can assume that the solar mass is 2×10^{30} kg, and that the gravitational constant $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$.

(ii) How do the radiation force and gravitational force on the electron behave as a function of the distance r from the sun (for $r > R_o$)?. What then is the equation of motion for $r(t)$, and what is its solution as a function of time, if the electron starts at a distance $r_o = r(t = 0)$ from the sun?

SECTION B: LONG QUESTIONS (ANSWER 2 of THESE)

QUESTION B.1: SUPERFLUID ^{4}He Superfluid 4He is the best known neutral superfluid; this question looks at some of its properties.

(i): Draw the energy dispersion relation for quasiparticles in superfluid ^{4}He (ie., the plot of the energy $\epsilon_{\mathbf{p}}$ as a function of the momentum p). Now, explain why it is that an object with mass $M \gg m_4$ (where m_4 is the mass of a ^{4}He atom), which is moving through superfluid ^{4}He, will move without friction until it reaches a critical velocity $v_c \sim min(\epsilon_{\mathbf{p}}/|\mathbf{p}|)$. You should consider the problem at finite T, where thermally excited quasiparticles already exist.

(ii) Suppose that the object of mass M were to be moving in some fluid along the $\hat{x}$ direction. Suppose also that this fluid has a constant viscosity coefficient η, so that there is a force $-\eta \mathbf{v}(t)$ acting on the particle in the direction opposite to its velocity $v(t)$ along $\hat{x}$. Find the equation of motion of the particle, assuming that there is also an external force $f(t)$ acting on it along $\hat{x}$.

Then show that if (i) this force is $f(t)$, and (ii) the initial velocity at $t = 0$ is $v(t = 0) = v_o$, then the solution to the equation of motion is

$$v(t) \;=\; v_o e^{-\gamma t} \;+\; \int_0^t dt' \, \frac{f(t')}{m} \, e^{-\gamma(t-t')}$$

where $\gamma = \eta/M$.

Now, suppose that the force is actually a constant in time, so that $f(t) \to f_o$. Show that after a long time has elapsed, the particle will then reach a constant terminal velocity v_f, and give the result for v_f in terms of f_o, γ, and M. How could you have very simply derived this result without solving the equation of motion?

(iii): In a superfluid things are a little different because the friction depends on the velocity. Suppose that the friction coefficient in the superfluid behaves with velocity according to $\eta(v) = \eta_o(v - v_c)\,\theta(v - v_c)$, where $\theta(x)$ is just the Heaviside or "step" function (so $\theta(x) = 0$ for $x < 0$, and $\theta(x) = 1$ for $x > 0$).

To solve the equation of motion here is complicated - but you can still find the new terminal velocity v_f without doing this. Find the result for v_f.

(iv): In a real superfluid one can also have quantized vortex ring excitations, which behave like quasiparticles in that they can also be excited by interactions with an external body. For a circular ring, one has approximately that the energy E of the ring and the momentum p of the ring depend on the radius R of the ring according to

$$E \;\sim\; \tfrac{1}{2}\rho\kappa^2 R \ln\frac{R}{a_o} \;\; ; \qquad\qquad p \;\sim\; \pi\rho\kappa R^2$$

where ρ is the superfluid density, κ the circulation quantum, and $a_o \sim 0.1\ nm$ is a vortex core radius.

If in analogy with the quasiparticle argument, we suppose that the critical velocity for formation of a vortex ring is given by $v_c \sim min(E/p)$, then show that for a superfluid in which R can be as large as you like (a superfluid moving past an infinitely large object), then $v_c \to 0$. Show also that if the superfluid is moving through a cylindrical tube of radius R_o, then v_c is finite, and give an expression for it.

Finally, noting that the velocity of the vortex ring excitation itself is given by $v = dE/dp$, find an expression for the velocity $v(R)$ as a function of the vortex ring radius, and sketch a graph of it.

QUESTION B.2: DIATOMIC GAS A diatomic molecule has 3 degrees of freedom, viz., translational motion of the molecular centre of mass, rotational motion abut the centre of mass, and vibrations in distance between the 2 atoms. We will treat these different degrees of freedom as being independent, ie., with no coupling between them. We assume the diatom is made from 2 atoms, each with mass m, and mean separation a_o.

(i): The moment of inertia of the rotating diatom is $I = 1/2ma_o^2$. We also suppose that the frequency of small harmonic oscillation of the distance x around the mean a_o between the atoms is ω_o.

Show that we can write the total canonical partition function $\mathcal{Z}$ for a gas of N such diatoms as $\mathcal{Z} = Z_{tr} Z_{rot} Z_{vib}$, where Z_{tr} comes from the translational degrees of freedom, where $Z_{rot} = z_I^N$ and $Z_{vib} = z_{\omega_o}^N$, and show that

$$z_I \;=\; \sum_{j=0}^{\infty}(2j+1)\exp[-\beta\hbar^2 j(j+1)/2I] \;; \qquad\qquad z_{\omega_o} \;=\; \sum_{n=o}^{\infty}\exp[-\beta\hbar(n+\tfrac{1}{2})\omega_o]$$

You do not have to evaluate the translational term Z_{tr}.

(ii) Let us first consider the vibrational modes. Evaluate the partition function $z_{\omega_o}(\beta)$, and then show that the vibrational contribution to the energy of the system is $U_{vib}(\beta) = \frac{1}{2}N\hbar\omega_o \coth(\beta\hbar\omega_o/2)$. From this find also the contribution $C_V^{vib}(\beta)$ to the specific heat.

Finally, sketch the behaviour of both $U_{vib}(\beta)$ and $C_V^{vib}(\beta)$ as functions of the temperature T.

(iii) Now let's look at Z_{rot} for the rotational motion of the diatom. The low T behaviour is easy, because the terms in the sum in the expression for $z_I(\beta)$ decrease rapidly with increasing j. By taking just the first 2 terms in the sum, find a simple low-T result for $z_I(\beta)$, and from this find expressions for $U_{rot}(T)$ and $C_V^{rot}(T)$ for the N diatoms in the low T regime.

For the high-T behaviour we need to approximate the sum as an integral. Using the result $\int_0^\infty dx\, x\, e^{-x^2} = 1/2$, find a simple result for $z_I(\beta)$ in the high-T regime where $kT \gg \hbar^2/2I$, with the result $\propto kT$. Then, from this result, find the energy U_{rot} and $C_V^{rot}(T)$ for the N diatoms in the high T regime.

Finally, plot sketches for U_{rot} and $C_V^{rot}(T)$ for the N diatoms as a function of T; you can use the expression you found for the low-T and high-T results, and then just simply interpolate between them.

(iv) The "third" contribution to the specific heat coming from the translational degrees of freedom is just that from a 3-dimensional classical Maxwell-Boltzmann gas. Typically, the vibrational zero point energy $\hbar\omega_o/2 \gg \hbar E_o$, where $E_o = \hbar^2/2I$ is the rotational zero point energy. Using the results you have derived above for $C_V^{rot}(T)$ and $C_V^{vib}(T)$, sketch the result you expect for the TOTAL specific heat $C_V(T)$ for a gas of N diatoms, as a function of T. Explain the limiting behaviour you find for $C_V(T)$ for (i) high T (ie., for $T \gg \hbar\omega_o/2$) and for low T (ie., for $kT \ll \hbar^2/2I$)?

QUESTION B.3: The YOUNG and the OLD UNIVERSE Near the beginning its life, the universe was composed of a variety of fermionic particles, plus photons. Near the end of its life (using extrapolations from what we already know), it will be a mixture of black holes and photons.

(i) Describe the universe as it was until a time $t = \tau_o$ after the Big Bang, where $\tau_o \sim 400,000$ yrs (you can ignore the time in the first few years after the Big Bang). What happened around $t \sim \tau_o$, and why? Why did this happen when the temperature $T \sim 4,000$K?

(ii) In the earlier stages of the universe (for $t \ll \tau_o$), we can assume that the system is ultra-relativistic, meaning that the fermion particle energy $\epsilon \gg mc^2$, where m is the fermion rest mass. We can also assume the system is at very high temperature, so that $|\mu|/kT \to 0$. Under these conditions, show that the energy of a fermion with momentum p is $\epsilon_p \sim pc$, and find expressions for (a) the number density $\rho = N/V$, and (b) the energy density $u = U/V$, for the fermions - showing in particular that $u \propto T^4$. You can write the answers in terms of the definite integrals

$$F_n = \int_0^\infty dx \frac{x^n}{e^x + 1}$$

which you do **not** need to evaluate.

(iii) In the universe at present (at a time $t \sim 1.4 \times 10^{10}$ yrs after the Big Bang), the universe is populated by a mixture of matter and radiation (plus the enigmatic "dark matter"). The matter is a mixture of stars and black holes, along with a lot of sub-stellar "junk" (sub-stellar brown dwarfs, planets, planetoids, dust, gas, etc.).

Most of the stars will end up as white dwarfs (which then cool to black dwarfs after a time period up to $\sim 10^{14}$ yrs). Why does this happen? Which stars will not end up as black dwarfs, and what will happen to them?

Low-mass black dwarfs can be treated using the usual non-relativistic Chandrasekhar argument, to show that their radius $R_o = cM^{-1/3}$, where the constant $c = 2C_f/GC_g$ is derived by minimizing the sum of the degeneracy energy $U_f = C_f M^{5/3}/R^2$ and the gravitational energy $U_g = -GC_g M^2/R$, where R is the black dwarf radius, and M is its mass. Suppose now that the dwarf has not yet cooled (ie., it is still white), so that there is an extra small radiative thermal contribution $U_T = \alpha(T, M)R^2$ to the energy. Assuming that $\alpha(T, M)$ is "small", find the new solution $\tilde{R} = R_o + \delta R$ for the radius, by looking for the small correction to the original minimization equation (formally, we assume that both δR and α are $\sim O(\epsilon)$, where $\epsilon \ll 1$, and isolate terms $\sim O(\epsilon)$ in our equations).

(iv) After extremely long times almost all matter will amalgamate into black holes, apart from a photon bath which steadily cools (after far longer times $> 10^{100}$ yrs, almost all of the black holes will decay by the Hawking process into radiation as well). Suppose at some given time the volume of the universe is V_H. Using the Planck result that the total photon energy of the universe $U_{ph} \propto V_H T^4$, show that the photon bath obeys $C_V(T) \propto V_H T^3$, and from this, assuming that the expansion of the universe is adiabatic, find the dependence of the entropy on V_H ad T.

Finally, let us assume that the expansion of the universe obeys the Hubble law, so that $V_H(t) \propto t^3$, and using the Planck result that the energy density of a photon gas is given in MKS units by

$$u(T) \;=\; \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}$$

find (a) the temperature of the photons after a time $t = 700 \times 10^9$ yrs, assuming that at the present time of $t = 1.4 \times 10^{10}$ yrs, one has $T = 2.7\ K$; and (b) find the photon energy density of the universe at the present time, in MKS units.

QUESTION B.4: METALS and INSULATORS Solids can be classified into metals or insulators. Very roughly speaking, we can say that (a) Metals have mobile electrons, with dispersion relation $\epsilon_p \sim p^2/2m_1$, a Fermi surface, and behave similarly to an electron gas, whereas (b) electrons in an insulator with energy near the Fermi energy are bound to atoms and not mobile, and have no Fermi surface. Only above an "energy gap" are they mobile, and we can write an approximate dispersion relation $\epsilon_p \sim E_o + p^2/2m_2$. The "masses" m_1 and m_2 are not necessarily equal to the free electron mass m_o. Typically the gap $E_o \sim 1 - 2\ eV$ in size.

(i) Noting we also have acoustic and possibly optical phonons, draw pictures of how you think the specific heat $C_V(T)$ will behave as a function of temperature T, for both metals and insulators, and explain why the different contributions have the temperature dependence that they do.

(ii) At low T, a degenerate fermion system shows a specific heat of form $C_V(T) \propto g(E_F)T$, where $g(E_F)$ is the 1-particle density of states at the Fermi energy. From this result, deduce the low-T behaviour of (a) the energy $U(T)$ (b) the entropy $S(T)$, and (c) the free energy $F(T)$. Can you give a qualitative argument which justifies the result you get for $U(T)$?

A useful way to measure the density of states $g(E)$ in a metallic system is to look at the rate of photon absorption by the metal as a function of photon frequency ω. Photons will only be absorbed if an electron can be excited from an occupied state at one energy to an unoccupied state at another higher energy. Draw what you think you would see for the photon absorption as a function of frequency ω in (a) a low T metal, and (b) a low T insulator, with $E_o = 2\ eV$.

(iii) A very common approximation when dealing with acoustic phonons is to assume a phonon density of states $g(E) = 9E^2/(k_BT_D)^3$ for $0 < E < \theta_D$, and $g(E) = 0$ for $E > \theta_D$. Here T_D is the "Debye temperature" and $\theta_D = k_BT_D$ is the "Debye energy". Typically T_D is somewhere in the range $100\ K - 600\ K$ for different solids.

From this information, you should be able to derive an integral expression for $\ln \Xi(T)$ for the acoustic phonon system (where Ξ is the grand canonical partition function), and also for the energy $U(T)$ [HINT: use the analogy with photons]. Assume a system of unit volume, so that $U(T) = u(T)$, the energy density; and assume that the atoms taking part in acoustic vibrations each have mass M. You do not need to evaluate the integrals over energy. You will use the result that the phonon chemical potential $\mu = 0$; why is this the case?

Finally, we want to evaluate the root mean square displacement of atoms in the solid caused by acoustic phonons. This can be shown to be given by $\bar{x} = [\langle x^2 \rangle]^{1/2}$, where

$$\langle x^2 \rangle \;=\; \frac{\hbar^2}{2M}\int \frac{dE}{E} g(E)[1 + 2n(E)]$$

in which $n(E)$ is the Bose distribution function. Derive an integral expression for $\langle x^2 \rangle$, and then show that in the low temperature limit $T \ll T_D$, we have a finite $\bar{x}$ given by

$$\bar{x} \;\sim\; \tfrac{3}{2}\hbar(1/M\theta_D)^{1/2} \qquad\qquad (T \rightarrow 0)$$

How do you interpret this result physically?

(iv) All of the above ignores the fact that in any real solid there will be defects (which behave like 2-level systems), electronic spin impurities, and nuclear spins. To isolate out the effect of electronic spin impurities in an insulator, we can apply a magnetic field. Suppose these impurities have spin-1/2, and we apply a magnetic field B to the system. What then is the partition function for a set of N such impurities, and what is their contribution to the specific heat? Finally, draw a graph of the resulting specific heat for an insulator in the range $0 < T < 50\ K$, assuming that (a) the Debye temperature $T_D = 500\ K$, and (b) the magnetic moment of the spin impurities is $\mu/k_B = 0.7\ K/T$, where T means "Tesla", and we are in an applied field of $20\ T$.

END of FINAL EXAM