PHYS 403: REVISION QUESTIONS: SET 3

(April 18th, 2021)

Here is another set of model exam questions for this course. Again, they will help you revise for the exam. Note that the actual exam will mix questions of different kinds (short vs long, and easy vs harder) together.

NB: The exam will be a take-home exam. It will begin at the time given in the exam schedule, but it will be a 30-hour exam. As already noted in an email to everyone, the exam will be structured so that you only have to answer half the questions - you will be able to choose which ones. You will have answer half of the short questions and half of the long questions.

QUESTION (1) BREEDING RABBITS: Let us imagine a large island, with lots of food, on which there lives a species of rabbit which can be either brown or grey. Each time a pair of rabbits breeds, it produces a litter of 6 baby rabbits. The young rabbits mature very rapidly - in roughly 1 month they can also breed. Assume (a) that the number of male and female rabbits is the same; (b) that each female breeds on average once each 60 days, and (c) that the lifespan of a rabbit is 10 years.

(i): If we start off with 100 young rabbits, then how many will there be after 5 years (1800 days)?

(ii): Suppose now we introduce a population of hawks, each one of which eats 1 rabbit per day. This then limits the growth of the rabbits - eventually, in less than 10 yrs, their population will equilibrate at a total population N. Suppose that $N = 10^4$; then what is the number n of hawks?

(iii): Suppose the hawks cannot distinguish between grey and brown rabbits. Once we have reached equilibrium, with 10^4 rabbits, then what is the probability that we will have m_g grey rabbits (where $0 < m_g < 10^4$)?

(iv): Suppose instead that the hawks can distinguish the 2 colours of rabbit, and that they prefer the taste of brown rabbits, so that they preferentially catch them; the probability that a hawk catches a brown rabbit is twice the probability that it will catch a grey rabbit. What then will be the probability that we will have m_g grey rabbits (where again, $0 < m_g < 10^4$)?

QUESTION (2) SIMPLE QUESTIONS ABOUT QUANTUM FLUIDS: Here are some **qualitative** questions about normal and superfluid quantum liquids. We Consider the drag forces acting on a small object being moved through either (a) an ordinary classical gas or a fluid, or (b) a Bose superfluid or (c) a normal Fermi fluid. One then notes the following:

(a) In the superfluid, below the transition temperature T_c, the drag is actually zero, provided the object moves with velocity less than some critical velocity. In an ordinary gas or liquid, and in the normal Fermi fluid, it is never zero - moreover it increases with velocity roughly like $F(v) \propto v$.

(b) Both for the case of the superfluid below T_c, and for the classical fluid or gas, the drag force at low velocities does not depend much on the temperature T. However in the normal Fermi fluid, one finds that $F(v) \propto T$ at low velocity (as well as being proportional to v).

Explain all of these observations as best you can. There is no need to derive results quantitatively; just explain the observations as described.

QUESTION (3) ENERGY FLUCTUATIONS: Let us start from the canonical partition function $\mathcal{Z}$ for a system $\mathcal{S}$.

(i) Show that the "mean energy squared" of the system $\mathcal{S}$ is given by $\langle E^2 \rangle = \mathcal{Z}^{-1}(\partial^2 \mathcal{Z}/\partial \beta^2)$.

(ii) Using this result work out an expression for the mean squared energy fluctuation in the energy of the system, written as $\langle \Delta E^2 \rangle = (\langle E^2 \rangle - \langle E \rangle^2)$; and then show that it can be written as

$$\langle \Delta E^2 \rangle \;\; = \;\; -\frac{\partial \langle E \rangle}{\partial \beta}$$

QUESTION (4) NUMBERS for FERMI SYSTEMS: Give numerical estimates for the Fermi energy (in eV), the Fermi temperature (in degrees K), and the Fermi velocity (in ms^{-1}), for the following systems, using the numbers given:

(a) Au in the metallic state (density $\sim 2 \times 10^4\ kg/m^3$, atomic weight ~ 200).

(b) the nucleons in a heavy nucleus (density $\sim 10^{18}\ kg/m^3$).

(c) liquid ^{3}He (atomic volume $\sim 60 \mathring{A}^3$ per atom).

To do these questions you need to learn to convert between energies in electrical (eV) and temperature (K) units.

QUESTION (5) VARIABLE HARMONIC WELL: Suppose we have a particle of mass m in a 1-dimensional potential well with length scale L which can vary; the potential $V(x)$ governing the motion is assumed harmonic, of form $V(x) = kx^2/2L^2$.

(i) Write down the harmonic frequency ω_L, the energy levels $\epsilon_n(L)$, and the partition function for this system; and from these find the energy $U(\beta, L)$ and specific heat $C_v(\beta, L)$ for this system. draw graphs of these as functions of temperature T.

(ii) Draw graphs of $U(\beta, L)$ and $C_v(\beta, L)$ as functions of temperature T for this system.

(iii) Now write down the free energy $F(T)$, and find the relation between the 1-dimensional "pressure" p, the 1-dimensional "volume" L, and the temperature T for the system (this is often called the "equation of state"; here it is defined in 1 dimension, and relates p, L, and T).

QUESTION (6) RELATIVISTIC GAS: Consider a 3-dimensional gas of indistinguishable particles in the very high-energy, low-density astrophysical limit, where the statistics is irrelevant - thus we are dealing with a 3-dimensional Maxwell-Boltzmann gas. We will assume that the energy and temperature are so high that the gas is relativistic, so that the energy of the particles goes as $E = \hbar ck$, where k is the wave-number of the particles (and $\hbar k$ the momentum).

(i) Find the partition function for a macroscopic system of N of these particles in a volume V, so that the density is $\rho = N/V$.

(ii) Find the energy U as a functions of T and N.

(iii) Find the entropy S, as a function of T, N, and V.

(iv) Now find the equation of state, ie., find the pressure p as a function of T and ρ.

END of THIRD SET of MODEL QUESTIONS