PHYS 403: REVISION QUESTIONS: SET 2

(April 11th, 2021)

Here is another set of model exam questions for this course. Again, they will help you revise for the exam. Note that the actual exam will mix questions of different kinds (short vs long, and easy vs harder) together.

NB: The exam will be a take-home exam. It will begin at the time given in the exam schedule, but it will be a 30-hour exam. As already noted in an email to everyone, the exam will be structured so that you only have to answer half the questions - you will be able to choose which ones. You will have answer half of the short questions and half of the long questions.

QUESTION (1) SET of HARMONIC OSCILLATORS: A D-dimensional oscillator (ie., a single oscillator oscillating in a D-dimensional space) can be thought of as a set of D independent 1-dimensional oscillators (because the motion in each different dimension is decoupled from the others).

(i): If the frequency of oscillation in each of the D dimensions is the same (an *isotropic* oscillator) and equal to ω_o, show that the possible energy eigenvalues of this system are $\epsilon_n = (n + D/2)\hbar\omega_o$. Then show that the degeneracy of the n-th eigenvalue is

$$g_n \;=\; \frac{(n+D-1)!}{n!(D-1)!} \tag{0.1}$$

(ii): Suppose we now have an *anisotropic* oscillator, so that the oscillation frequency is each direction is different, and we have oscillation frequencies ω_k, with $k = 1, 2, \cdots D$, for the different directions.
What now are the eigenvalues of this D-dimensional oscillator? And under what circumstances will the degeneracy of all the different eigenvalues be unity, ie., we will have no degeneracy for any of the levels?

(iii): Now find the partition functions for a set of N of these oscillators, both for N identical isotopic oscillators each with frequencies ω_o, and a set of N anisotropic oscillators, each with frequencies $\{\omega_k\}$.

(iv): Finally, find results for the total energy $U(T)$ and the specific heat $C_V(T)$ for both the isotropic and anisotropic N-oscillator systems.

QUESTION (2) SINGLE HARMONIC OSCILLATOR: Suppose we have a 1-dimensional oscillator with energy levels $\epsilon_n = (n + \frac{1}{2})\hbar\omega_o$, where ω_o is the frequency of the oscillator.

(i) Show that the partition function for this oscillator is $\mathcal{Z}(\beta) = \frac{1}{2} cosech\,(\hbar\beta\omega_o/2)$, where $cosech\, x \equiv 2/(e^x - e^{-x})$.

(ii) Using this result, find an expression for the mean energy U of the system. Then draw a rough graph of this result for the energy U as a function of $x = k_BT/\hbar\omega_o$.

QUESTION (3) QUALITATIVE PHYSICS: Be quantitative if you can - but the answers also require qualitative physical understanding.

(i): Why does the entropy of a pair of systems usually increase when one combines them physically into a single system? When does it not increase? If one then takes a single system, and separates it into two systems, the entropy also increases - why is this?

(iI): Suppose I have two sets of spin-1/2 systems, each containing N spins in an applied magnetic field which splits each spin level by an amount $2\Delta_o$. We assume that they are in every way identical except that one of the N-spin systems is at temperature $T = 0$, while the other is at temperature $T = \infty$.
Now, I combine the 2 sets of spins. What is the final energy of the combined system? And if I do the combination in an irreversible way, what do you think is the final temperature?

QUESTION (4) STRETCHED STRING: An elastic string is of equilibrium length L and mass ρ per unit length when in equilibrium, is stretched from its equilibrium length by an infinitesimal length dl. It is found that the work required to do this is $dW = kldl$ where l is the extension (ie., the total length is now $L + l$).

(i) Find expressions for the total energy U and the free energy F of the string; and for the change dF in the energy under a change dl in the length.

(ii) Now find expressions for the change in entropy S of the string when stretched at constant temperature, and the change in energy of the system at constant entropy

QUESTION (5) N 2-LEVEL SPINS: We go to our usual set of N non-interacting 2-level systems here, in a magnetic field so that the energy of each spin is $\epsilon_j = 0$ (spin down) or $\epsilon_j = \Delta_o$ (spin up)

(i) Find the entropy $S(E)$ as a function of the total energy E of the system.

(ii) Derive Stirling's approximation for the factorial $n!$, and use it to find an approximate expression for the entropy $S(E)$. Draw a graph of $S(E)$, and determine the maximum and minumum values of this function. Find in the same way an expression for the mean energy U of the system for a temperature T.

(iii) Now suppose we allow the spins in this system to "delocalize", ie., to fly free and become a 3-dimensional Fermi gas of spins. Note that this gas is not just a set of particles; each particle also has two internal states, so you need to account for both the translational degrees of freedom and the internal degree of freedom. Find an expression for the canonical partition function of this system, both in the low-T degenerate Fermion regime (where $T \ll T_F$), and in the high-T Maxwell-Boltzmann regime.

(iv) Draw a graph of the how you now expect the entropy $S(E)$ of the system to vary with E? Why can the localized gas have a negative temperature?

END of SECOND SET of MODEL QUESTIONS