PHYS 403: REVISION QUESTIONS: SET 1

(March 28th, 2021)

The following is a set of model exam questions for this course. They will help you revise for the exam. Note that the actual exam will mix questions of different kinds (short vs long, and easy vs harder) together.

NB: The exam will be a take-home exam. It will begin at the time given in the exam schedule, but it will be a 30-hour exam.

QUESTION (1) CANONICAL PARTITION FUNCTION: Suppose we have a system $\mathcal{S}$ which is immersed in a thermal bath $\mathcal{B}$ which is at temperature T, such that energy can pass between $\mathcal{S}$ and $\mathcal{B}$; but no matter can pass between them. If the system $\mathcal{S}$ has allowed energies E_j, then the **canonical** partition function is $\mathcal{Z}(\beta) = \sum_j e^{-\beta E_j}$, where $\beta = 1/k_B T$ is the inverse temperature.

(i) The mean energy of the system is given by $U = \sum_j p_j E_j$, where p_j is the probability that the state with energy E_j is occupied. Show that $U = \langle E \rangle = -\mathcal{Z}^{-1}(\partial \mathcal{Z}/\partial \beta)$.

(ii) Suppose the system $\mathcal{S}$ is a single 2-level system, with possible energies $E_\pm = \pm \Delta_o$. Write down an expression for the partition function $\mathcal{Z}$ for this system, and show that the mean energy at temperature T is given by $U = -\Delta_o \tanh \beta \Delta_o$.

QUESTION (2) BLACK BOXES:

(i) Suppose I surround the sun (considered to be a black body at a temperature T_s) with a perfectly black shield. Assuming the shield is indestructible, it will come to an equilibrium temperature T_B, governed by the radiative energy flow from the sun and the energy radiated by the shield. Assuming that the universe outside the shield can be treated as a reservoir at temperature $T = 0$, show that the equilibrium temperature of the shield is $T_B = (T_s/2)^{1/4}$. Assume the shield is close to the sun's surface, so that the surface area of the shield is the same as that of the sun.

(ii) Now suppose instead I replace the single shield with 2 concentric shields. What is the temperature T_1 of the first (inner) shield, and T_2 for the second outer shield?

(ii) Finally let us generalize the argument to a set of N concentric black shields. What is the temperature T_n of the n-th shield, for $1 \geq n \geq N$?

QUESTION (3) FREE ENERGY: For a gas of particles, the infinitesimal changes dS in the entropy and dV in the volume of the gas container result in a change dU in the energy, given by $dU = TdS - pdV$, where T is the temperature and p the pressure in the gas.

(i) The free energy of the system is $F = U - TS$. Find an expression for the infinitesimal dF, and show that the pressure is then given by the partial derivative $p = -(\partial F/\partial V)|_T$, where T is held constant.

(ii) Suppose we are allowed to add particles to the gas as well, so that $dU = TdS - pdV + \mu dN$, where μ is the chemical potential of the gas particles and N is their total number in the container. Assuming again that $F = U - TS$, find an expression for μ in terms of a partial derivative of F; make sure to specify what is held constant.

QUESTION (4) EARTH'S ATMOSPHERE:

(i) Suppose we assume that the atmosphere is at a temperature T, so that we can describe the system by a canonical distribution. Show that the density of the atmosphere at a height z above sea level must then be given by $\rho(z) = \rho_o \exp -mgz/k_B T$, where ρ_o is the density at sea level.

(ii) Now re-derive this same result by considering an infinitesimal element of the atmosphere at a height z, and balance the gravitational force pulling it down against the pressure keeping it up; you can assume that the ideal gas law is obeyed so that $p = \rho K_B T$.

QUESTION (5) N-SPIN SYSTEM: Consider a set of N non-interacting spin-1/2 systems, in a magnetic field so that each of their energies are $\epsilon_j = \pm\Delta_o$

(i) Suppose the system is in equilibrium at temperature T; what is the energy $U = \langle E \rangle$ of the system?

(ii) Find the entropy $S(U)$ of the system as a function of U, again assuming equilibrium.

END of FIRST SET of MODEL QUESTIONS