PHYS 403: HOMEWORK ASSIGNMENT No. 2: CANONICAL and GRAND CANONICAL ENSEMBLES

(Feb. 13th, 2021)

NEW DEADLINE for HOMEWORK: MONDAY, March 15th, 2021

To be uploaded by 11.59 pm, March 15th- Late Homework will not be accepted

QUESTION (1) DISTRIBUTION of OSCILLATORS: Suppose we have a set of N non-interacting quantum oscillators, with different oscillator frequencies $\{\omega_j\}$. The oscillator frequencies are distributed equally in the range $0 < \omega_j < \Omega_o$, ie., the probability that an oscillator will have a frequency ω in this range is the same for any ω, and therefore independent of ω. Assume that N is very large, so that we can treat this probability as a continuous function of ω.

1(a) Find expressions for the free energy $F(T)$ and the energy $U(T)$ in terms of T, N, and $E_o = \hbar\Omega_o$.

1(b) Now find an expression for the specific heat $C_V(T)$ of the system. Once you have done this, find limiting expressions for this result, in the 2 limits $k_BT \gg E_o$ and $k_BT \ll E_o$.

QUESTION (2) EQUILIBRIUM for HYDROGEN GAS: Consider a H atom inside a 3-d box with side L. When $L \to \infty$, in free space, the bound state levels for the electron, labeled by quantum number n, have energy $E_n = -E_o/n^2$, where $E_o = me^4/8\epsilon_o^2 h^2$ is the Rydberg constant ($E_o \sim 13.605\ eV \sim 157,870\ K$ in temperature units).

When L is finite, we an estimate the maximum value of n by noting that the the radius r_n of the n-th state is $r_n \sim r_o n^2$, where $r_o = 4\pi\varepsilon_o\hbar^2/me^2$ is the Bohr radius. For the atom to fit into the box, we then require that $r_{max} = L$, giving a condition for the maximum value n_L for some value of L.

We will use this toy model to derive some rather important approximate results. We will ignore all states with energy $E > 0$, ie., all excited states of the system, and only consider the bound states.

2(i) First consider the case $L \to \infty$, and calculate the canonical partition function $Z(T)$ and the energy $U(T)$. Compare the results for $T > 0$, and for $T = 0$.

2(ii) Now consider a finite box with $L = r_n$, so there are only n_L bound-state levels in the box. Find expressions for $Z(T)$ and $U(T)$; then approximate these results for $n_L \gg 1$ by continuous integrals over n between the limits $1 < n < n_L$. One can estimate these integrals for both the case $n_L^2 \gg E_o/k_BT$, ie., for $T \gg T_L$, and the case $T \ll T_L$; here $k_BT_L = E_o r_o/L$. Give these estimates, and try to interpret them for a gas of atomic hydrogen at density $\rho = 1/L^3$.

Hint: to fully understand the result you find in 2(i) it will help to also do 2(ii), and then compare the result with what you found in 2(i).

QUESTION (3) ELECTRON-POSITRON GAS: We ionize a H gas of number density n_H, at very high T (ie., $k_BT \gg E_o$, where E_o is the Rydberg ionization energy). Ignoring the protons, we ask what are the number densities $n_+(T)$ and $n_-(T)$ of positrons and electrons at a temperature T. The positrons exist because at high energies, electrons can scatter off each other to create electron-positron pairs, in processes like $2e^- \to 2e^- + (e^+ + e^-)$ (note that high-energy proton-electron scattering will also produce $e^+ - e^-$ pairs).

3(i) In the interior of stars on can assume that $2mc^2 \gg k_BT \gg E_o$, where m is the electron mass. Now find $n_+(T)$ and $n_-(T)$ as a function of T. By assuming the chemical potentials μ_+ and $-\mu_-$ to be equal (ie., $\mu_+ + \mu_- = 0$), and assuming that because $2mc^2 \gg k_BT$, we have $n_+ \ll n_H$, use charge conservation to find this number density $n_+(T)$ of positrons.

3(ii) Now let's assume conditions that are reasonable near the centre of a star. Assume that the H density is $\rho_H = 100 g/cm^3$. Convert this to a number density, and find out at what temperatures one has number densities of (a) $n_+ = 10^{10}/cm^3$ and (b) $n_+ = 1/cm^3$.

Finally, assume that $T = 15$ million K (roughly the temperature at the centre of the sun), and find n_+.

END of 2nd HOMEWORK ASSIGNMENT