PHYS 403: HOMEWORK ASSIGNMENT No. 5: BOSE and FERMI GASES

(Mar 22, 2020)

HOMEWORK DUE: MONDAY, APR 6th, 2020 by 11.59 pm

To be sent in the Canvas website - Late Homework will not be accepted

QUESTION (1) BOSE GAS: This one is to help you with the various manipulations.

1(a) The energy of a spinless 3-dimensional Bose gas is given by $U = \sum_j g_j P_j E_j$, where g_j is the degeneracy of the j-th state, P_j the probability of occupation of this state, and E_j the energy of this state. Rewrite this in the form of an energy integral, incorporating both the density of states, and the Bose distribution function, as functions of energy.

1(b) Now show from this that when the temperature $T < T_c$, the Bose condensation temperature, the energy of a system of unit volume is given by the expression

$$U = \frac{2\pi}{h^3}(2mk_BT)^{3/2}k_BT \int_0^\infty dx \frac{x^{3/2}}{e^x - 1} \tag{0.1}$$

and write a final explicit expression for this energy using the result that the integral in this equation is given in terms of the Riemann zeta function $\zeta(z)$ and the Gamma function $\Gamma(z)$ by

$$\zeta(z+1) = \frac{1}{\Gamma(z+1)} \int_0^\infty dx \frac{x^z}{e^x - 1} \tag{0.2}$$

where z is real, and you can use the result that for integer n, $\Gamma(n + 1/2) = \frac{(2n)!}{4^n n!}\pi^{1/2}$.

1(c) Finally, using your result for $U(T)$, or by some other means, find an explicit expression for the specific heat $C_V(T)$ of this system (using $\Gamma(n + 1) = n!$ for integer n).

QUESTION (2) PHOTON GAS: For the photon gas, the chemical potential $\mu = 0$ for all temperatures. We can find the grand thermodynamic potential for this system by then treating it as Bose gas with a density of states appropriate to a linear dispersion relation with frequency $\nu = ck/2\pi$, and the corresponding density of states .

2(a) Show, using an integration by parts, or in some other way, that the grand canonical partition function Ξ is given by the expression $\ln \Xi = AT^3$, and find the constant A.

2(b) Now, using this result, find an expression for the specific heat $C_V(T)$ of the photon gas.

QUESTION (3) FERMI GAS: There are many things that one can ask here, about either the degenerate Fermi gas or the high T Fermi gas. This question will focus on sometings that I have not covered in detail.

3(a) Using numbers that you can easily find or determine for yourselves, for densities and masses of the particles in the relevant systems, give numerical estimates for the Fermi energy of (i) electrons in a typical metal (ii) nucleons in a heavy nucleus or a neutron star, and (iii) ^{3}He atoms in liquid ^{3}He (for which the atomic volume is 4.62Å^3 per atom). You should give your answers in either eV units, or temperature units; note that $1eV = 1.60219 \times 10^{-19}$ J, and $k_B = 1.3807 \times 10^{-23}\ J/K$, so that $1eV \sim 11,604$ K.

3(b) Then consider the special case of Al metal, where the electrons behave like a pretty good Fermi gas. Show that if the electron concentration is $\sim 1.81 \times 10^{29} m^{-3}$, then the electrons at the Fermi energy (ie., at the Fermi surface) have a "Fermi velocity" $v_F \sim 2.01 \times 10^6$ m/s.

CONTINUED ON NEXT PAGE...

3(c) We are now going to establish some general relations which apply to both Fermi and Bose gases of free particles. First, show that in the grand canonical ensemble that the following relations hold:

$$d(k_B T \ln \Xi) \;=\; N d\mu + S dT + p dV; \qquad\qquad pV \;=\; k_B T \ln \Xi \tag{0.3}$$

and then show that

$$p \;=\; k_B T (\partial \ln \Xi / \partial V)|_{\mu,T} \;\;=\;\; -\sum_j \frac{1}{e^{\beta(E_j-\mu)} \pm 1} \frac{dE_j}{dV} \;\;=\;\; \frac{2U}{3V} \tag{0.4}$$

where in the distribution function, ± 1 applies to the Fermi and Bose gases respectively.

3(d) Finally, using this last result for the pressure of a Fermi gas, estimate the Fermi energy and the pressure for a gas of electrons of density $10^6 g/cm^3$ confined to a sphere with radius 6×10^3 km (ie., the size of the earth); these are typical numbers for a white dwarf star.

END of 5TH HOMEWORK ASSIGNMENT