PHYS 403: HOMEWORK ASSIGNMENT No. 4: GASES and PLASMAS

(March 5th, 2020)

HOMEWORK DUE: MONDAY, MAR 16th, 2020

To be handed in during class- Late Homework will not be accepted

QUESTION (1) DENSITY OF STATES - RELATIVISTIC GAS: This is to give you some feeling and practise for calculating densities of states. In the book and in the notes, you learn how to find the density of states for a non-relativistic gas in 1, 2 and 3 dimensions; for this system, the energy dispersion goes like $\epsilon_k = \hbar^2 k^2/2m$.

Consider now a relativistic gas, where the dispersion goes instead like $\epsilon_k = ([m^2c^4 + \hbar^2 c^2 k^2]^{1/2} - mc^2)$. Using the same method of counting states that is done in 3-d for a non-relativistic gas, show that when we are in the extreme relativistic limit, where $\epsilon_k \gg mc^2$, the density of states for a set of spinless particles is given by

$$g_{3D}(E) \rightarrow \frac{1}{2\pi^2\hbar^3}\frac{E^2}{c^3} \qquad\qquad (\epsilon_k \gg mc^2)$$

QUESTION (2) ASTROPHYSICAL PLASMA: Here we get some experience on using the grand canonical partition function. We will consider an astrophysical plasma which contains a mixture of H and He atoms, along with the singly-ionized species H^+ and He^+, all in contact with a gas of electrons (some of which are liberated when these ionized species are produced by ionizing the atoms). We will assume that the ionization energy of H is $I_H = 13.6\ eV$, and that of He is $I_{He} = 24\ eV$. We will assume that when neither the H nor the He is ionized, we have N_H Hydrogen atoms and N_{He} Helium atoms per unit volume.

In the numerical part of this question you can assume that $1\ eV = 11,604\ K$, so as to convert energies between electron Volt and degrees Kelvin units; and assume the electron mass is 9×10^{-31} kg. You can also assume that the activity $\alpha = e^{\beta\mu}$ of a low-density electron gas is $\alpha(N) = \rho/\rho_q$, where ρ_q is the "quantum density" for the gas, given by

$$\rho_q = 2(mk_BT/2\pi\hbar^2)^{3/2}$$

2(a) Find the grand canonical partition function Ξ, and also the equilibrium atomic densities ρ_H, ρ_{He}, for the system of H and He (so that $\rho_H \rightarrow N_H$ when none of the H atoms are ionized, and likewise for the Helium). Write these results in terms of the chemical potential μ, the inverse temperature β, and the energies I_H, I_{He}, and assume that the energy of each ionized species is zero. To work this out you should use the fact that the grand canonical partition function Ξ_N for some gas of N atoms and their associated ions is just $\Xi_N = \Xi_1^N$, where Ξ_1 is the grand canonical partition function for a single atom and its associated ion.

2(b) Now suppose that we are in a 'molecular cloud' in which (i) the electron density is $10^9\ m^{-3}$; and (ii) the temperature $T = 50\ K$. Find the chemical potential μ, and the equilibrium fractional densities $n_H = \rho_H/N_H$ and $n_{He} = \rho_{He}/N_{He}$ for the H and He atoms in the cloud.

2(c) Now plot the chemical potential $\mu(T)$, and the same fractional densities $n_H(T), n_{He}(T)$, as a function of temperature in the range $10 < T < 10^4\ K$. Do this numerically, using the results you found in 2(a). You can plot it either as a function of T or of $\log T$.

END of 4th HOMEWORK ASSIGNMENT