REMARKS on GREEK MATHEMATICS

What is known of the history of Greek mathematics spans the period from Thales, around 600 BC, to the end of the 2nd Alexandrian school, around 300 AD. Enormous advances were made from 600-300 BC. The most notable early achievements were achieved by the school associated with Pythagoras in c. 550 AD, and then some 250 yrs later by the school associated with the great library in Alexandria, a city founded by Alexander the Great at the mouth of the river Nile (the library was put in place by Ptolemy, the successor to Alexander in Alexandria). However the entire period is marked by an extraordinary creative flowering, nourished by several different mathematical schools- over this time period these led to huge advances. Much of this work was lost during the middle ages, and some of it was not to be surpassed until the 19th century. The early work had a very large influence on philosophy- it is unfortunate that the ideas of Archimedes came too late to have much influence, since they opened the door to quantitative mathematical physics. However by this time the great Hellenistic intellectual outpouring was in decline, to be overtaken by a very different kind of regime, that of the Roman empire. The Romans had little use for Greek mathematics- the first to translate Euclid into Latin was Boethius, in 480 AD, some 800 years after Euclid wrote his famous "*Elements*". One can hardly refrain from noticing the symbolic import when a Roman soldier killed Archimedes- in retrospect this spelt the end of the Greek enlightenment.

(1) BRIEF HISTORICAL NOTES

(i) Pythagorean School: Pythagoras was born around 570 BC on the island of Samos, in Ionian Greece just off the coast of Asia Minor (what is now the Turkish coast), near the coastal towns of Miletus and Ephesus. It is said by some that he was a student of Thales of Miletus. Thales himself was apparently quite a gifted scientist, who apart from predicting a solar eclipse, also was noted for having measured distances of ships at sea (see also notes on pre-Socratics). Pythagoras left Asia Minor, apparently because of the policies of the tyrant Polycrates, and travelled widely, eventually settling in Croton, in Southern Italy. The school he founded there was very influential (notably on Plato); adherents wore a star pentagram symbol, and were distinguished by a strongly mystical philosophy. The school was initially very influential in Crton and environs, but after a revolt led by Cylon, in which a number of the school were killed, Pythagoras fled to Metapontum, and the school was reconstituted- however it was later persecuted and the survivors were dispersed as far afield as Thebes (southern Egypt), Phleius, and Tarentum. A split appears to have occurred at some point between 2 branches of the sect- the *acusmatiki* and the *mathematiki*. We do not have much direct evidence of their activities (much of what we know comes from Plato, Aristotle, and their students), since the sect appears to have been somewhat secretive in both its activities and the arcane knowledge and ideas it possessed. All this should again alert us to the utterly different culture we are dealing with here- instead of a group of armchair mathematicians in a university lab, we are talking about a secret religious sect with various hidden political and social aims, as well as a variety of rituals and taboos. One should get used to this in trying to understand thinkers at the beginnings of science- Newton was hardly different in his approach to Nature.

From what we know of Pythagoras he appears to have been one of the greatest intellects of the Ancient world, perhaps of all time, held in some awe by later Greek thinkers- his influence on Plato and Aristotle, some 150 yrs later, was enormous. His philosophy fused Orphic religious traditions with a belief in rational inquiry into the nature of the *kosmos*. The latter was viewed by him as *alive*, a living creature. For Pythagoras, the cosmos was a whole, without *telos* (end); we are all part of it (at least our souls are). The Pythagoreans believed in transmigration of souls- meat was forbidden, as were beans, since they might conceal the soul of a former friend (at least according to later parodies). Philosophy is the attempt to study and understand the cosmos- and in the end, philosophy is assimilation of the pupil to the divine cosmos. The *kosmos* represented for Pythagoras a kind of inherent order or structural perfection, a divine pattern or form. This was a remarkable new idea which broke completely from the Milesian school, which was essentially materialist and interested in the constituents of matter- in the long term it became one of the most important ideas in human history.

The essential tool for this study was mathematics- which revealed the form of the cosmos. Numbers were viewed as divine, the key to the *kosmos*, and their properties thereby revealed patterns in the cosmos. Most notable of these was the harmonic property, also revealed in music- this led to the whole idea of the music of the spheres. Pythagoras discovered the relation between musical harmony and simple fractional properties of musical intervals- it seems likely that he understood the relationship to frequencies of musical notes, for he argued the motion of the planets, and the speeds of the sun, moon, and stars, corresponded to the musical ratios of the 4th (ratio of 4/3), the 5th (3/2) and the octave (2/1). These were later extended to a set of 8 intervals by Plato. The idea seems to have been that the motion produce sounds that were impossible for us to hear.

The living cosmos had no end but it did have a beginning- it began with a 'seed' in the infinite, which was afire in its centre- this proceeded to grow by drawing in the infinite from outside itself, and giving it structure by numbers. This process of drawing in is called 'inspiration' (ie., breathing in). Later neo-Pythagoreans like Philolaus even argued that the earth was a planet revolving around this eternal fire- a remark which was picked up 2000 yrs later by Copernicus (we come to this in later notes).

The mathematical discoveries by Pythagoras came initially out of a study of the many properties of numbers, of their ratios, and of sequences and series of them. They were interested in the relationship to physical properties- another discovery credited to them is the understanding of the ratio known as the 'Golden ratio'. The discovery of irrational numbers by Pythagoras (ie., numbers that cannot be written as a fraction) was a great shock, since it proved not that geometrical figures could not be understood solely in terms of integers or their ratios- and led to the conclusion that magnitudes must be continuous. For a discussion of the proof that $\sqrt{2}$ must be irrational, see the course slides. This discovery led to many discussions of the paradoxes coming from infinite series of numbers, most notably those of Zeno.

A modern physicist, looking at how Pythagoras anticipated some of the later ideas in physics and mathematics, can only be quietly amazed.

Alexandrian schools: The 1st Alexandrian school had its heyday around 300 BC; Euclid, Archimedes, and Appolonius were the most distinguished teachers at the famous library there (essentially the world's first university), and they trained many students. The spirit of this institution was summed up by Euclid's famous retort when asked what use his theorems were... "give him 3 pennies, since he must make profit out of what he learns". Euclid (c. 330-275 BC) is famous for his book, the "*Elements*", which was not about the elements in the sense used by Empedocles or by modern chemists- it was instead a systematic treatment of most of the mathematics known at that time.

What is crucial to this work is the way the subject was developed, using rigorous proofs based on the axiomatic method. This was an attempt to develop in a logical way the essential elements of mathematics as then understood, starting from a basic set of propositions- it is discussed in more detail below. In the discussion of geometry these led to proofs of a large number of properties of simple or complex geometric shapes, both in 2 and 3 dimensions (this is why the work is so long). The work deals with everything from the 'theory of proportions', taken from the remarkable mathematician Eudoxus- this approach anticipates work even up to the 19th century in analysis- to number theory, and thence to geometry. The "Elements" must be one of the most widely used books of all time- it was still part of the core curriculum of high-school mathematics in many European high schools in the early 20th century! Euclid was not the only contributor to geometry from this first Alexandrian school- indeed Appolonius (262-200 BC) was a pioneer in the study of 3-d solid geometry, and responsible for the the theory of conics and conic sections (ellipses, parabolae, hyperbolae).

The other great representative of the 1st Alexandrian school was Archimedes. It will be noticed that in all the work of the pre-Socratic Greeks, as well as that of Plato and Aristotle, on the nature of the material world, no quantitative discussion appears anywhere. Even Democritus never tried to analyze, say, the volume of objects built up from different-shaped atoms; and the ideas of Plato and Aristotle never get anywhere near to quantitative discussion. From this point of view the work of Archimedes on physical problems was quite remarkable- although his contemporaries payed much attention to their applications (military and otherwise), the style and method of his investigations, and the results, were too far ahead of their time to be properly appreciated. It was not until Galileo that a similar combination of mathematics and empirical investigation was brought to bear on the physical world.

Some well-known examples of his work are the understanding of static mechanics (ie., the calculation of the forces acting on static bodies, their centres of gravity, etc), and of hydrostatics (the laws governing the displacement of fluids, and forces exerted by them). The famous discovery of the relation between quantity of displaced fluid and weight reduction of an immersed body, in his bath, is part of this work.

Unfortunately the changes wrought by the rise of Roman militaristic power were not conducive to disinterested inquiry, and it was not until the brief flowering of the 2nd Alexandrian school, around 300 AD, that further important steps were made. In this later flowering of research, mathematicians such as Pappus (c. 300 AD) or Diophantus (c. 320 AD) continued the development of geometry - although Diophantus was more concerned with understanding the abstract properties of numbers- what later Arab mathematicians called *algebra*. After this, the collapse of Western European civilisation meant that almost all of these ideas were lost, except in the Arab world. During the middle ages Arab mathematicians made important advances, and also preserved many of the written works of the Greeks (although much was lost in the fire that destroyed the great library in Alexandria, with the loss of 700,000 volumes). Without this Arab lifeline to the Renaissance, it is hard to imagine how the modern world would have turned out.

(2) MATHEMATICAL IDEAS & ACHIEVEMENTS of the GREEKS

(i) Number Theory: Just a brief word here. The development of numbers went to some extent in parallel with the development of astronomy, and the roots of these are buried in prehistoric times. This is hardly surprising- it is known that a number of birds and mammals are capable of counting, and it is obvious that prehistoric man could do this. What was crucial was the development of a written notation for keeping track of numbers (ie., accounting), and for manipulating them. These skills were already possessed by the Babylonians and Assyrians, but it is perhaps surprising how much depended on having an adequate notation to describe numbers and their operations. From the appropriate slide one can see how this notation developed, but the history is long and tortuous. It is worthwhile noting that the refinement of mathematical notation goes on today, and has often been associated with key advances in the subject. For more on this subject, see some of the references (in "supplementary material").

The interest of the Greeks in the properties of numbers began with the Pythagorean school, and with Democritus, who were interested in the extraordinary variety of properties numbers and collections of them can display. Amongst other things this led to an interest in irrational numbers, and how to approximate them, but gradually a very sophisticated understanding of number theory was built up. By the time of Plato, extensive results were known- important figures being Theodorus, Theatetus, and Eudoxus. however things really got going with the later Alexandrian schools. The advances eventually made by the Greeks in our understanding of number theory (still one of the most difficult and subtle branches of mathematics) were staggering. This work was driven purely by curiosity but led to discoveries of enormous importance. The most notable figures in all of this were Pythagoras, Archimedes, Euclid, and, much later, Diophantus, who invented what is now a whole field in modern mathematics (the field of 'Diophantine equations'). Some of this work is of considerable interest today, and it was essential to the revival of mathematics in Europe during the renaissance (Thus Fermat's famous "last theorem" of 1637, one of the most famous theorems in mathematics, and only very recently proved by Wiles, was found inscribed in Fermat's copy of the book *Arithmetica*, by Diophantus; this book was apparently carried everywhere by Fermat). There is no space here to describe all this work- those who are interested can go to the references.

(ii) Series, Approximations, and the beginnings of Calculus: The example given in the slides (Democritus's method of finding the area of a triangle, which he also generalized to find the volume of a cone) is useful because simple. More complicated examples of note were the problem of 'squaring the circle (ie., finding out π), and of 'doubling the cube' (ie., finding out $^3\sqrt{2}$). However it is important to stress that this sort of thinking led to 2 very important developments:

(a) The idea that successive approximation to some quantity could give a kind of "limiting operation". Even if the approximation never terminated, if successive terms ever more closely approximated the correct answer, then the approximation was useful. In the same way one could construct an infinite series which summed to a finite value (eg., the series $S = 1 + 1/2 + 1/4 + ...1/2^n + ...$, which in the limit of an infinite number of terms gives the simple answer $S \rightarrow 2$). The use of such series and approximations became particularly important to the Greeks once the existence of irrational numbers was understood, (a fraction does not need such manoeuvres for its evaluation, whereas an irrational one does). This is not to say that the Greeks accepted infinite series equably- the paradoxes of Zeno discussed in the slides show that they worried a lot about them- but they learnt how to deal with them.

(b) An important step was taken when attempts were made to evaluate the areas of *curves* (as opposed to simple straight line figures). This is a much harder problem, but we have seen in the slides how it was done for the circle, by dividing it into successively smaller triangles (leading to the approximate evaluation of π). In the hands of Archimedes, the techniques of dividing areas and volumes into increments, making these successively smaller, and then calculating results for areas, volumes, and centre of gravities, were developed into a fine art, with strict proofs for all the results. In this sense Archimedes invented the integral calculus, although in a less streamlined form than the later renaissance re-invention.

(iii) Greek Geometry & the Axiomatic Method: The axiomatic method was an attempt to develop in a logical way the essential elements of mathematics as then understood, starting from a basic set of propositions. These basic propositions are called '**axioms**'. An axiom is a single proposition which is simply assumed, without proof. All theorems of an axiomatic system are then derived from a few axioms using the rules of the system (the 'rules of inference').

The idea in the axiomatisation of geometry is that one specifies the relations between a set of primitive entities by the axioms (in the case of geometry these are points and lines), ie., by a set of rules. At the same time as fixing the rules of operation with the primitive entities, one can also provide an *interpretation* of the system, by specifying how to relate the objects/entities to objects in the real world- such as geometrical figures, in the case of geometry. However, what was crucial to the whole exercise, is that this latter step is *not necessary.* In other words, one can deal with propositions about points and lines, derived using the axiomatic method from a primitive set of propositions (the axioms), without having the slightest idea what these words and propositions might refer to.

The imaginative leap to such an abstraction was quite prodigious. In the 21st century we can think about it as follows- imagine that the axioms are provided to a computer, in the form of a set of instructions about what operations are allowed on a set of primitive identities which the computer can call 'birds' and 'bees' (but which we might instead like to call 'points' and 'lines'). The crucial (and perhaps non-intuitive) thing is that any 'meaning' of the terms "point" or "line", and all statements about them, is entirely acquired via the axioms. The 'meaning' nowadays is usually taken to refer to an *interpretation* of the basic entities and of the theorems in the logical system, by connecting them to objects either in some larger logical system, or to objects in the real world. But all we really have in this logical system is the theorems- this is just the set of all propositions in the logical system that can be derived from the axioms using the rules of inference. These propositions are 'true' if they can be derived from the axioms, and false if their contrary can be derived. Thus true means 'derivable from the axioms'.

If we then wish to interpret these as objects in the real world (eg., as lines, points, etc., in 3-d space) then we are at liberty to do so. However, in the modern view, the only way of deciding if this interpretation is a correct one (in the sense that true propositions about objects in the logical system are also true of the corresponding objects in the real world) is by experimentally checking in the real world. If this correspondence is valid, then we can if we wish talk about the meaning of the propositions in the logical system as though they really corresponded to statements about the real world.

As a measure of what an enormous step this was, we can look at what happened to one of the features discovered by Euclid that perplexed him. This is the story of the famous "5th axiom", or the "axiom of parallels". This axiom states that

"Given a straight line, and a point not on the line, there is one and only one line passing through this point which is parallel to the first line."

Another way to put this is that there is only one line through the point which will not cross the first line, if they are both extended to infinity. What bugged Euclid, and hundreds of mathematicians for 2000 years after him, is that this axiom seemed on the face of it to be unnecessary. The proposition seemed so self-evidently true of real figures that it was hard to see how it didn't follow from basic axioms defining straight lines, points, parallel, etc. And yet Euclid took a rigorous approach and found that he could not prove it from these other axioms- it was necessary to add it as an independent axiom. Thus arose the famous problem- how to find a proof that the 5th axiom did follow from the others. Many such 'proofs' were devised, but they were all incorrect.

Finally, in 1829, the shattering conclusion was published by Lobachevski, and independently in 1832 by Bolyai (although it seems Gauss had already made the discovery in 1824). These mathematicians took the bold step of *denying* the 5th axiom, and showing that one could get not one but an infinite variety of other "non-Euclidean" geometries. Finally, in the 20th century, Einstein capped it all by showing that the universe was indeed described by non-Euclidean geometry! We get to this later, in discussing relativity. But the point to be emphasized here is that Euclid, in developing the axiomatic method, and sticking to it, had opened the door to possibilities which could not have been arrived at from observations of the world, or intuition based on experience.

At the time Euclid did this work, the implications were too far-reaching for the Greek world. In fact the work of Pappus 600 years later was still further developing Euclidean geometry; and Descartes, Newton, Laplace, Maxwell, etc., never questioned it as a description of the real world.

(3) GREEK ASTRONOMY

Greek astronomy had its roots in the much older work done in the Baylonian and Egyptian civilisations. As mentioned before, the Babylonians already had extensive astronomical records, with good measurements of time and postions in the sky (from which we inherit both our systems of angular and time measurement- the 360^o circle and the day of 24 hrs, 60 minutes, and 60 seconds). Thus the prediction of a solar eclipse by Thales relied on these earlier observations, and on the preliminary understanding of them already achieved.

The Greeks of the golden age were aware of some astronomical phenomena. They had even been involved in some of this. Thus, eg., Anaxagoras was aware that the moon shines by reflected light, and gave a theory of lunar eclipses which recognised the spherical shape of the earth and its consequent shadow at the moon; he also recognised the equivalence of the morning and evening stars as one planet (Venus). However it is equally clear from reading Plato and Aristotle that they were very far from understanding the implications of what was known. This came later, mainly with the 1st Alexandrian school of mathematicians. A colleague of Archimedes at the Alexandrian library (and who became one of its first directors) was Eratosthenes of Cyrene (now Aswan, in southern Egypt). By measuring the difference between shadows cast by the sun in Alexandria and Cyrene, and knowing the distance between them, he estimated the diameter of the earth- by then understood to be spherical. His answer was accurate to roughly 1 per cent (The much later result of Ptolemy was less accurate, to only 2 per cent- his result was a distance of 29.5 earth diameters, and the correct answer is 30.2 diameters). Aristarchus, another contemporary of Eratosthenes, went much further- indeed, he advanced pretty much the complete Copernican hypothesis- arguing that the sun was the centre of the solar system, and that the planets, including the earth, revolved around it in circular orbits. According to Archimedes this caused him a little trouble (there was an attempt to indict him by a certain Cleanthes), but nevertheless the hypothesis was adopted by his successor Seleucus. Aristarchus also attempted to measure the distance of the sun, obtaining a result of 180 earth diameters (the correct answer is 11,726 earth diameters). These measurements were done by triangulation, ie., measuring the position of the sun in the sky at 2 widely separated places at the same time- their accuracy was of course limited by the accuracy of measurement of the sun's position, and of time.

It might be thought at this point that the Greeks were well ont he road to developing modern astronomy (albeit without instruments). Unfortunately subsequent developments were characterised by increasingly accurate observations, but increasingly less accurate ideas to explain them- again, this can be understood in terms of the changing intellectual climate, which was not favourable to speculative theorising. Thus the influential Hipparchus (161-126 BC) wrote systematically on trigonometry, discovered the precession of the equinoxes, measured the lunar month and produced an accurate star Atlas with positions of 850 stars, measured the lunar month to 1 second accuracy, and measured the distance of the sun as 1245 earth diameters. His contemporary Posidonius (Cicero's teacher) obtained the even more accurate value of 6,545 earth diameters. However Hipparchus also rejected the Aristarchean hypothesis of a solar-centred planetary system, and instead espoused the 'epicycle theory' (invented by Appollonius in 220 BC). The epicycle system is discussed more fully in our section on Copernicus, later in the notes- it later became (c. 150 AD) to be known as the Ptolemaic system, after its most influential supporter.

The story after this is of a gradual decline in understanding, accompanied by an increasingly sophisticated effort in observation and measurement. By the time Ptolemy produced his *Almagest* in the 2nd century AD, Alexandria had long been an integral part of the Roman empire, which had little use for speculative philosophy or what we would now call 'theoretical physics/astronomy'. The ideas of Aristarchus and his more daring colleagues had been almost forgotten, to be subsumed by an unwieldy system which although much less radical, was in fact incapable of explaining observations in real detail without endless and arbitrary elaboration. We return to this in the notes 1300 yrs later in history, when we come to Copernicus.

The history of Greek astronomy and mathematics is, like that of Greek phiosophy, something of a Greek tragedy- heroic efforts in the early days, which carried the network of theoretical understanding in a wave almost to modern times- followed by a recession of the wave, back to an intellectual straightjacket which stifled the development of ideas until the Renaissance. There are perhaps 2 main reasons for this. The first was simply that the tides of war and history took away just as easily as what they had given- once the Romans came to power the Greek ideas and methodology became mere curiousities, to be lost with the later demise of the Roman empire.

The second reason is more subtle. The Greek work in philosophy, physics, and astronomy was too far ahead of its time, and moreover lacked a crucial element. Often it is said that this was the lack of experiments, systematic observations, and the experimental methodology that came much later. But this is only very partly true- indeed, Alexandrian astronomy was breathtakingly accurate in its later stages, and almost obsessed with data-collecting and measurement. In fact the problem was deeper, at the theoretical level. Greek physics and astronomy were, almost inevitably, slaves of the philosophical framework forced on them by the Athenian philosophers (which itself reflected ideas going back to Pythagoras, Heraclitus, and Parmenides). The prevailing orthodoxy was essentially a *geometric* one, overly influenced by considerations of symmetry and abstract mathematical form. Totally lacking was a place for *mechanics*; there were no forces, no gravity, no understanding of dynamics in the modern sense. Concepts like the "void", or vacuum, or of 'force' in the more modern sense, of particles or atoms, were largely ignored because they did not fit the standard model. The models of Aristotle in philosophy, or of Ptolemy in astronomy, conformed to this idea of *what a theory should be.* The ideas of Leucippus, Democritus, and Archimedes, did not- and so although they did get a hearing, their influence was much less durable.