GREEK MATHEMATICS

The history of Greek mathematics spans the period from Thales, around 600 BC, to the end of the 2nd Alexandrian school, around 415 AD. Enormous advances were made from 600-200 BC. The most notable early achievements were from the school associated with Pythagoras in c. 550 AD. Then some 250 yrs later (essentially in the period 300-200 BC), came the amazing achievements of the '1st Alexandrian school'. This school was associated with the great library in Alexandria (the most important of the many cities founded by Alexander the Great, this one at at the mouth of the river Nile - the library was put in place by Ptolemy, the successor to Alexander in Alexandria). This entire period is marked by an extraordinary creative flowering, nourished by several different mathematical schools- over this time period these led to huge advances. The 3 best known of the remarkable mathematicians of the 1st Alexandrian school were Archimedes, Euclid, and Apollonius; the first two of these, along with Pythagoras, are amongst the greatest mathematicians in history. Much of their work, along with that of Eratosthenes, Aristarchus, Hero, and Hipparchus, was lost during the early middle ages, and some of it was not to be surpassed until the 19th century.

The early work of the Pythagorean school had a very large effect on philosophy- it is unfortunate that the ideas of Archimedes came too late to have much influence, since they opened the door to quantitative mathematical physics. However by this time the great Greek intellectual outpouring was in decline, and never again acquired its first lustre. The Greek world was to be overtaken eventually by a very different kind of regime, that of the Roman empire. The Romans had little use for Greek mathematics- the first Roman to translate Euclid into Latin was Boethius, in 480 AD, some 800 years after Euclid had written his famous "*Elements*". One can hardly refrain from noticing the symbolic import when a Roman soldier killed Archimedes- in retrospect this spelt the end of the Greek enlightenment.

Nevertheless there was a second flowering of Alexandrian mathematics, beginning in roughly 30 BC; although not encouraged by the Romans, it was also left unhindered. The best-known mathematicians of the 2nd Alexandrian school were the remarkable Diophantus, who invented the branch of number theory now bearing his name, plus Pappus and Hypatia (daughter of Theon, himself also a member of this school). The 2nd Alexandrian school is also notable for the very influential astronomical work of Ptolemy. Mathematics at this later date was immersed in a rather different philosophical climate, that of the neo-Platonists and neo-Pythagoreans, whose effect on mathematical research was not terribly constructive. The 2nd Alexandrian school was brought to a sad close with the rise of the Eastern Christian church, which was very hostile to any kind of learning - with the murder of Hypatia by the Christians in 415 AD, the great period of Greek mathematics effectively ceased. A number of the Alexandrian students of Hypatia did manage to make their way to Athens, amongst them Proclus, and for a brief time this 'Athenian school' of mathematics held on- but it was finally closed, at the instigation of the Christians, by a decree of Justinian in 529 AD. By this time most of the original 700,000 works in the great library at Alexandria had been destroyed by the Christians - one of the greatest acts of intellectual desecration in history. The job was completed in 641 AD, with the invasion of Alexandria by Arab armies only 9 years after the death of Mohammed. It is said that the remaining volumes in the library took 6 months to burn.

In what follows we will (1) look at the historical evolution of Greek mathematics, and then (2) examine some of the key ideas and achievements that came out of it. In a separate chapter I will give a brief discussion of the advances in Astronomy that were being made in parallel with those in mathematics.

(1) BRIEF HISTORICAL NOTES

1(A) Early Greek Mathematics (600-300 BC)

It was the mathematical work of the early Greek mathematicians which had such an influence on Plato- he learnt most of this from his contemporaries and friends Theatetus, Eudoxus, and Archytas (there is no evidence that Plato himself did any important original work).

We have very little of the original writings of the ancient mathematicians- just as with the philosophers, we need to rely on the commentaries of later writers, and on fragments of the originals. In Ancient times the writings of earlier mathematicians were widely available, and so historians have been able, by a process of cross-referencing, to give a sketchy picture of how things developed. This is particularly important for mathematics done before 430 BC. Some of the main sources include a commentary of Proclus, written roughly 450 AD, which was mostly occupied with discussions of Euclid, but also relied heavily on an earlier history of mathematics, now lost, written by Eudemus around 350 BC; this and other sources tell us quite a lot of what was in Eudemus's history. There are also fragments of a "General View of Mathematics" written by the Roman Geminus around 50 BC; and by various means one can infer what were the earlier sources of many of the propositions in Euclid's works (themselves written around 300 BC), which we have in essentially complete form. With all this, the following picture emerges:

(i) IONIAN SCHOOL: The life of Thales (c. 640-550 BC), the first of this school, was already recounted in the section on the pre-Socratic philosophers. In mathematics and astronomy we are aware of 2 main contributions. First, a set of geometric propositions concerning triangles and circles- the most interesting of these is the demonstration that the angle subtended by a diameter of a circle, as seen from any point on the circle not on the diameter, is a right angle (it is said that he sacrificed an ox after achieving this result). Second, the prediction of a solar eclipse which took place on May 28, 585 BC. The success of this prediction gave Thales a reputation all over the Greek world, and indeed ensured his name in history. The irony is that there is no way he could have made such a prediction with complete confidence- this would have required a knowledge and understanding of planetary dynamics far beyond the Greeks (indeed, such predictions only became possible well after Newton). What is most likely is that Thales was aware of the extensive Egyptian and Chaldean records of solar eclipses, extending over many centuries, where a pattern had been noted of the frequent repetition of solar eclipses at intervals of 18 years and 11 days (NB: frequent, but by no means always- whether a solar eclipse occurs depends on a variety of factors, and is complicated by the eccentricity of both the lunar and terrestrial orbits).

Thales taught mathematics and astronomy to a number of his pupils- it is widely believed that Anaximander (611-545 BC) was one of these, and he himself had a lively interest in astronomy (as already noted earlier, in our discussion of the pre-Socratic philosophers). Anaximander also is thought to have introduced the '*gnomon*' (also called the '*style*') to the Greek world- this was a stick or post stuck vertically into the ground, and used both as a sundial, and with considerable geometrical insight, as a means of deducing the meridian, the dates of the solstices, the inclination of the ecliptic, and finally the latitude of the gnomon itself. There is evidence that Anaximander indeed measured the latitude of Sparta in this way, and doubtless other places.

(ii) PYTHAGOREAN SCHOOL: This school was of great importance, not only because of its great impact on subsequent philosophy, but because of the real mathematical achievements of Pythagoras and his followers.

Pythagoras: Pythagoras was born around 570 BC on the island of Samos, just off the coast of Asia Minor- he may therefore have met Thales, and it is believed that at some time was taught by Anaximander. For further details of his life, see the section on pre-Socratics.

According to Pythagoras, the essential tool for the study of the cosmos was mathematics- which revealed its form. Numbers were divine, and their properties therefore revealed patterns in the cosmos- most notably the harmonic property, revealed in music, and in the 'music of the spheres'. As already noted, his remarkable philosophy, according to which the universe had to be understood in terms of basic 'forms', related to mathematical intervals, was very influential in later centuries. This was also true of his mathematics. We can group his basic results into 2 categories:

Theory of Numbers: The mathematical discoveries by Pythagoras came initially out of a study of the many properties of numbers, of their ratios, their factors, and of sequences and series of numbers. A pivotal discovery for Pythagoras was his discovery of the relation between musical harmony and simple fractional properties of musical intervals- it seems likely that he understood the relationship to frequencies of musical notes, for he argued that the motion of the planets, and the speeds of the sun, moon, and stars, corresponded to the musical ratios of the 4th (ratio of 4/3), the 5th (3/2) and the octave (2/1). These were later extended to a set of 8 intervals by Plato. The idea seems to have been that the motion of the planets produces sounds that were impossible for us to hear. It is likely that he also understood the geometric patterns revealed by crystals in a similar way.

Pythagoras also initiated the study of what are called 'polygonal numbers', in his study of triangular numbers- his school later studied the higher polygonal numbers. Triangular numbers are numbers that can be represented by a set of counters arranged in the form of a right isosceles triangle, with each row containing one more counter than the previous one, up to a row of n counters- the triangular number is the total number of counters. Thus in modern notation, a triangular number T_n can be written as the sum $T_n = \sum_1^n m \ = 1 + 2 + 3 + ... + n$, ie., $T_n = n(n+1)/2$. The polygonal numbers are defined analogously, in terms of counters arranged in the shape of some polygon. The interest of triangular numbers for Pythagoras lay in the connection to the geometric properties of triangles (such as that expressed in the famous theorem of Pythagoras- see below). In investigations like these, Pythagoras and his followers discovered quite a few geometric analogues of results for numbers, which could be expressed in the form of different sorts of sums over numbers, or their squares, etc. It is clear that the remarkable patterns that emerged were a source of great inspiration to them, both in finding new mathematical results, and in their broader philosophical ideas.

The discovery of irrational numbers by Pythagoras (ie., numbers that cannot be written as a fraction) then came as a great shock, since it proved that geometrical figures could not be understood solely in terms of integers or their ratios. For a discussion of the proof that $\sqrt{2}$ must be irrational, see the course slides. The shock here was not only a mathematical one- it was also philosophical, since as already noted, Pythagoras felt that the numerical properties of natural phenomena (revealed in shapes, musical tones, etc.) were fundamental to the underlying form of the cosmos. The effects of this discovery were therefore profound. In the first instance it apparently led to the divorce of the study of numbers from that of geometry and measurement in the real world. One can therefore argue that this discovery, and its interpretation by the Pythagorean and later philosophical schools, played a very unfortunate role in separating Greek thinking on philosophy and mathematics from their more 'scientific' investigations of the natural world. This is a theme I will return to again later. The discovery of irrational numbers also influenced later discussions of the paradoxes coming from infinite series of numbers, most notably those of Zeno the Eleatic (see 'pre-Socratics')

The beginning of the clarification of this mess had to wait until after the invention of the calculus, in the late 17th century. From work by Cantor at the end of the 19th century, we now know that the density of irrationals on the line is infinitely greater than that of the rational fractions, even though both are infinitely dense- a thorough study of all of this leads deep into set theory, to some extraordinary properties of infinitesimals, and to the properties of 'infinity' (and there infinitely many different types of infinity). All this happened in the late 19th and early 20th centuries, and will be discussed later on.

Plane and Solid Geometry: It is clear that Pythagoras and his later followers were much preoccupied with the relation of numbers to plane and solid geometry, in various ways. The music of the spheres, the relations between different regular solids, and planar figures, were for them merely a hint of deeper structures in the cosmos. They were also interested in the relationship to physical and even aesthetic properties- another discovery credited to them is the understanding of the ratio known as the 'Golden ratio'. One of the more controversial questions here is to whom can be attributed the discovery of the 5 regular solid figures discussed by Plato (these are solid geometrical figures whose vertices all lie on the surface of a sphere). There is good evidence that Pythagoras may have already found all of these, although Plato attributes the discovery that the octahedron and icosahedron were in this class to his contemporary and friend Theaetetus, and the cube, pyramid, and dodecahedron to Pythagoras.

Most of the geometric work of Pythagoras centred on the study of planar geometric figures (triangles, parallelograms, circles, etc), and many of the results in the first 2 volumes of Euclid's 11-volume work (and some in the 5th book) were first found by the Pythagoreans. The most famous of these results is of course 'Pythagoras's theorem', taught in all schools, according to which the length c of the hypotenuse of a right triangle is related to the lengths a and b of the other 2 sides, as $a^2 + b^2 = c^2$. The proof later related by Euclid is certainly not the one used by Pythagoras, and various ways that he may have proved it have been suggested by historians.

Of the later Pythagoreans, only Archytas, a contemporary of Plato, contributed anything really notable. He was best known in antiquity for having solved the then notorious 'Delian problem' (named because it is said to have originated in a prophecy from the oracle of Apollo at Delos), that of 'duplicating the cube' (ie., finding the ratio between the sides of 2 cubes whose volume differs by a factor of 2). He was a native of Tarentum, and had great political influence there and elsewhere in the Greek world- at one time he used this influence to save the life of Plato

The Pythagorean school initiated 2 enormously important developments in mathematics. The first was the style of rigorous deduction in mathematics- although such deduction had existed previously, the first attempts to make it systematic appear with Pythagoras. The 2nd great development was the attempt to link the study of numbers with that of geometry, and the systematic study of the property of numbers, to launch what we now call number theory. A modern scientist, looking at how Pythagoras influenced later ideas in physics and mathematics, can only be quietly amazed. The proof of the existence of irrational numbers is one of the greatest landmarks in the history of mathematics, embodying not only the discovery of a huge new mathematical territory, but also a very novel style of deductive argument.

(iii) ATHENIAN SCHOOL: This sub-section is unfinished, and will be inserted later

A major difficulty had arisen in mathematics by the time of Eudoxus, viz., the problem of irrational numbers, understood at that time as the fact that certain lengths were not comparable. The method of comparing two lengths x and y by finding a length c so that $x = mc$ and $y = nc$, for some whole numbers m and n, had been shown to fail for certain pairs of lines of certain lengths (such as the example of $x = 1$ and $y = \sqrt{2}$ given by Pythagoras).

1(B) 1st Alexandrian School (300-30 BC)

The 1st Alexandrian school had its heyday from roughly 300-200 BC; Euclid, Archimedes, and Apollonius were the most distinguished teachers at the famous library there, and they trained many students. The library at Alexandria was founded by Ptolemy, who had been one of Alexander's generals and closest colleagues during his famous campaigns- after Alexander's death, Ptolemy seized control of the Egyptian part of Alexander's empire. In founding the library, in 313 BC, Ptolemy founded what was essentially the world's first university, for he not only began to amass a huge collection of ancient manuscripts from all over the Alexandrian world (amounting already to 600,000 manuscripts only 40 years later), but also brought some of the finest scholars to Alexandria to work and teach there. The result was that the best mathematicians moved to Alexandria, from Athens and elsewhere. The spirit of this institution was perhaps summed up by Euclid's famous retort to a student when asked what use his theorems were... "give him 3 pennies, since he must make profit out of what he learns". Although the 1st Alexandrian school continued until about 30 BC, its greatest creative period effectively came to a close with the death of Archimedes in 212 BC.

Euclid (c. 330-270 BC): Not much is known about the life of Euclid- indeed, what we do know has been a subject of considerable debate. Euclid is most famous for his "*Elements*", which was not about the elements in the sense used by Empedocles or by modern chemists- it was instead a systematic treatment of most of the mathematics known at that time. Nowadays one can buy the entire *Elements* in heavily annotated editions from various publishers around the world- Euclid however divided it into 13 books, each dealing with different aspects of mathematics. It is certainly one of the most influential books in human history- the style of proofs given in it were central to mathematics for over 2000 years, and of overwhelming influence in the mathematics and philosophy of the Renaissance. It was still being used as a standard text in most schools and even universities in Europe until the beginning of the 19th century, and in the UK until the early 20th century, and many modern teachers of mathematics lament its passing. Its historical importance was summarized by the scientific historian van der Waerden:

"*Almost from the time of its writing and lasting almost to the present, the* 'Elements' *has exerted a continuous and major influence on human affairs. It was the primary source of geometric reasoning, theorems, and methods at least until the advent of non-Euclidean geometry in the 19th century. It is sometimes said that, next to the Bible, the* 'Elements' *may be the most translated, published, and studied of all the books produced in the Western world.*"

What is crucial to this work is the way the mathematical results were developed, using rigorous proofs based on the axiomatic method. This was the first comprehensive attempt to develop, in a logical way, the essential elements of all of mathematics as then understood, starting from a basic set of propositions. The basic idea of the axiomatic method is discussed in more detail in section 2 below. In the discussion of geometry these led to proofs of a large number of properties of simple or complex geometric shapes, both in 2 and 3 dimensions (this is one reason the work is so long). The work deals with everything in planar geometry from simple properties of trangles, parallelograms, etc., to the 'theory of proportions' (taken from the remarkable mathematician Eudoxus- this approach anticipates work even up to the 19th century in analysis); all of this, plus many applications, appear in books 1-6. In books 7-10 he goes on to number theory, dealing in book 10 with irrantioan numbers (this latter seems to have been based on work of Theatetus and others going back to Pythagoras, as well as modifications introduced by Euclid). Finally in books 11-13 he deals with solid geometry, finishing in book 13 with a discussion of the 5 'Platonic Solids', and of Eudoxus's 'method of exhaustion'. The most brilliant parts of Euclid are to be found mostly in the geometric sections of his work, where the power of the logical development is shown at its most elegant. Euclid was not the only contributor to geometry from this first Alexandrian school- indeed Apollonius (262-200 BC) made far greater original contributions a short time after him. However Euclid's *methodology* was to be crucial to the development of mathematics for over 2000 years, as well as having a central role to play in some parts of Western philosophy.

For those interested in the details of the '*Elements*', there is a set of Supplementary Notes on this work. Note that this was not the only work of Euclid that has survived - his work on Optics, for example, was of considerable importance, giving the first understanding of perspective (even the the physical parts of it are incorrect!). A couple of other more minor works have survived, including one on astronomy, but unfortunately it seems that some rather major works are lost, including a 4-volume work on conics, a treatise on music, and a work entitled '*Book of Fallacies*', which according to Proclus dealt in a colloquial way with reasoning in everyday life, by

"*enumerating in order the various kinds* [of Fallacies], *exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of the error with practical illustration.*"

We shall see the influence of Euclid over and over again as we proceed through the development of modern physics. His influence in the 17th century is shown by Newton's reluctance to develop the results in his '*Principia*' in terms of the calculus which he had used to discover them- instead everything was developed in Euclidean fashion, using geometric proofs. In the 19th century the relaxation of Euclid's famous 5th axiom led to the development of non-Euclidean geometry. In the early 20th century this led Einstein to his greatest work- the General Theory of relativity, which unified our understanding of non-Euclidean spacetime with gravity and matter.

Archimedes (287-212 BC): Archimedes was a native of Syracuse, son of the astronomer Phidias, and he spent much of his life there- his association with the 1st Alexandrian school comes because he was taught there, and was in regular touch with the school during his life. His closest associate in Alexandria was Conon of Samos, a friend with whom he regularly corresponded. Quite a lot is know about his life- he was rather well-known in the Greek world for his inventions, which were plentiful and in some cases very useful (the 'Archimedes screw' is still being used to pump water in some parts of the world!). His best-known inventions in ancient times were the lever, the compound pulley, the catapult, and the use of parabolic mirrors to focus light, which he used to set ships on fire. In Plutarch's '*Lives*', he describes 2 of these inventions. First, the use of the lever and pulley:

"*Archimedes had stated that given the force, any given weight might be moved, and even boasted, we are told, relying on the strength of demonstration, that if there were another earth, by going into it he could remove this. Hieron being struck with amazement at this, entreated him to make good this problem by actual experiment, and show some great weight moved by a small engine. He thereupon fixed upon a ship of burden out of the king's arsenal, which could not be drawn out of the dock without great labour and many men; and, loading her with many passengers and a full freight, sitting himself the while far off, with no great endeavour, but only holding the head of the pulley in his hand and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea.*"

In 215 BC the Roman general Marcellus attacked Syracuse, and Archimedes was enlisted by King Heiron II to help defend the city. His use of mirrors to set the ships on fire was so effective that the Romans apparently had to abandon their ships and set about a lengthy siege, which lasted 3 years, hindered as it was by the use of catapults by the defenders. Plutarch's description of Archimedes's role in the defense includes the following passage:

"*When Archimedes began to ply his engines, he at once shot against the land forces all sorts of missiles, and immense masses of stone that came down with incredible noise and violence, against which no man could stand; for they knocked down those upon whom they fell in heaps, breaking all their ranks and files. Meanwhile huge poles were thrust out from the walls over the ships, and sunk some by great weights which they let down from on high upon them; others they lifted up into the air by an iron hand or beak like a crane's beak and, when they had drawn them up by the prow, and set them on end upon the poop, they plunged them to the bottom of the sea; or else the ships, drawn by engines within, and whirled about, were dashed against steep rocks that stood jutting out under the walls, with great destruction of the soldiers that were aboard them. A ship was frequently lifted up to a great height in the air (a dreadful thing to behold), and was rolled to and fro, and kept swinging, until the mariners were all thrown out, when at length it was dashed against the rocks, or let fall.*"

When Syracuse was finally taken, Marcellus gave orders that at all costs, Archimedes was to be spared. However a soldier apparently found him in the midst of a geometric demonstration being written in the sand- when Archimedes ignored the soldier, he was executed by him. According to the stories, Marcellus personally accorded the same treatment to the soldier in question.

The stories about Archimedes would hardly be complete without recounting how, during a visit to a public bath-house, his idea for what is now called Archimedes's theorem came to him- according to doubtless apocryphal sources, he cried "*Eurekaa*" and ran naked through the streets to his house, in order to elaborate the proof.

Archimedes left behind a considerable number of works, which include: (i) "*On plane equilibria*" (two books which deal with static mechanics- in particular, with the centres of gravity of a large number of planar objects including parabolic sections); (ii) "*Quadrature of the parabola*", then (iii) 2 books entitled "*On the sphere and cylinder*", which deal with the volumes of spheres and cylinders, and of portions sliced off them; (iv) "*On spirals*, which gives a thorough discussion of the geometry of spirals, and areas of sections enclosed by them; (v) "*On conoids and spheroids*", which exmines the volumes of sdegments of solid bodies, including paraboloids of revolution, hyperboloids of revolution, and spheroids obtained by rotating an ellipse either about its major axis or about its minor axis; (vi) The famous 2 books entitled "*On floating bodies*", which develop hydrostics, and include 'Archimedes's theorem, on the weight of a body immersed in water, and the related displacement of water; (vii) The "*Measurement of a circle*" which calculates the area of a circle by a process of inscribing ever finer triangles in it; and (vii) "*The Sandreckoner*", in which he proposes a number system capable of expressing very large numbers (up to 8×10^{16} in modern notation). He argues that this number can count the number of grains of sand which could be fitted into the volume of the universe- thereby giving his estimate of the size of the universe. Finally (ix), a most remarkable work, which in the last few decades has come to be known as the "*Palimpsest*" of Archimedes (a palimpsest is a manuscript whcih has been subsequently covered by another one). The original title is "*The Method*", and in it Archimedes describes how he gets some of his geometric results, including the role of mechanical demonstration to help him to the answer, before he finds a proof. This work was found in 1906, by the Danish philologist Johan Ludvig Heiberg, in the library of The Church of the Holy Sepulchre in Istanbul. The history of the wanderings of the Palimpsest over 2300 years, to its auction in 1998 to an anonymous bidder at Christie's in New York for $2 million, is quite extraordinary, and is recounted in a Supplementary Note.

From out point of view, the 2 most important achievements of Archimedes were (i) His laying of the foundations of what later became the calculus, by enormously extending the ideas of Eudoxus on the 'method of exhaustion'; and (ii) the first work in mathematical physics. The latter is of particular interest to us here. It will be noticed that in all the work of the pre-Socratic Greeks, as well as that of Plato and Aristotle, on the nature of the *material world*, almost no quantitative discussion appears anywhere. Even Democritus never tried to analyze, say, the volume of objects built up from different-shaped atoms; and the ideas of Plato and Aristotle never get anywhere near to quantitative discussion. From this point of view the work of Archimedes on physical problems was quite remarkable- although his contemporaries payed much attention to their applications (military and otherwise), the style and method of his investigations, and the results, were too far ahead of their time to be properly appreciated. It was not until Galileo that a similar combination of mathematics and empirical investigation was brought to bear on the physical world.

It would seem that Archimedes himself had a similar view about the importance of his purely scientific work, as compared to the inventions that made him famous. As Plutarch says:

"*Archimedes possessed so high a spirit, so profound a soul, and such treasures of scientific knowledge, that though these inventions had now obtained him the renown of more than human sagacity, he yet would not deign to leave behind him any commentary or writing on such subjects. Instead, repudiating as sordid and ignoble the whole trade of engineering, and every sort of art that lends itself to mere use and profit, he placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life- studies, the superiority of which to all others is unquestioned, and in which the only doubt can be whether the beauty and grandeur of the subjects examined, of the precision and cogency of the methods and means of proof, most deserve our admiration.*"

Thus Archimedes understood perfectly well what was important in his work, and was under no delusions about how his *oeuvre* was regarded by most of his contemporaries. As we shall see, not much has changed since then- the obsession of modern media (TV, newspapers, etc), politicians, and business, is still in the short-term, particularly military, applications of research in physics and the other sciences. In many countries today (eg., Canada, which has never had any government policy in science and technology, nor anyone in government who has known anything about science) there is almost no understanding whatsoever of the role that pure research has to play in the longer-term evolution of society. This is curious, given that the advance and even survival of many of the great civilisations in the past has depended crucially on their development of new science- and things are no different today.

Apollonius (262-200 BC): Apollonius was a pioneer in the study of 3-d solid geometry, and responsible for the the elaborate and well-nigh complete theory of conics and conic sections (ellipses, parabolae, hyperbolae). His work is of less direct importance to us, but it was famous in the Greek and Arabic worlds, and was a triumph of the Euclidean method of exhaustive geometric proof. The famous work "*Conics*" came in 8 books, of which 7 have survived (4 in the original Greek, and 3 others from Arab translations); at least 6 other works were written by Apollonius, some of which have come down to us from the Arabs.

Apollonius was born in Perga, which today is known as Murtana and is in Antalya, in Turkey. Perga was an important cultural centre at this time, and was also the place of worship of the Greek goddess Artemis (who herself certainly originated from farther East). When he was a young man, Apollonius went to Alexandria where he studied under the followers of Euclid; he later taught there himself. His character was later contrasted with that of Euclid (who was reputed to be very modest and equable); Apollonius himself was apparently a rather difficult person.

I do not discuss the geometric work of Apollonius here because although it is of great interest to histroians of mathemtics and of Greek thought, it has had little role to play in the evolution of physics. The same is not so true of the history of astronomy (a subject which is now a sub-branch of physics, but which for most of its history was in important ways independent of physics). Apollonius was an important figure in the history of Greek astronomy, and he along with others used geometrical models to explain planetary theory. Ptolemy in his book "*Syntaxis* credits Apollonius with the introduction of systems of eccentric and epicyclic motion, to explain the apparent motion of the planets across the sky. This is actually incorrect- Eudoxus certainly had done some of this beforehand- but it shows to what extent Ptolemy was influenced by Apollonius. We note here that the strength of Greek interest in geometry was decisive in the eventual Ptolemaic formulation of a theory of planetary motions and of the Solar System- this Ptolemaic system, described in more detail in section 2 below, was only much later overthrown by Copernicus.

(c) 2nd Alexandrian School (30 BC - 415 AD)

Unfortunately the changes wrought by the rise of Roman militaristic power were not conducive to disinterested inquiry, and it was not until the brief flowering of the 2nd Alexandrian school, around 300 AD, that further important steps were made. In this later flowering of research, mathematicians such as Pappus (c. 300 AD) or Diophantus (c. 320 AD) continued the development of geometry - although Diophantus was more concerned with understanding the abstract properties of numbers- what later Arab mathematicians called *algebra*. After this, the collapse of Western European civilisation meant that almost all of these ideas were lost, except in the Arab world. During the middle ages Arab mathematicians made very important advances, and also preserved many of the written works of the Greeks (although much was lost in the fire that destroyed the great library in Alexandria, with the loss of 700,000 volumes). Without this Arab lifeline to the Renaissance, it is hard to imagine how the modern world would have turned out.

this sub-section on the 2nd Alexandrian school is unfinished- it will be done later

(2) MATHEMATICAL IDEAS & ACHIEVEMENTS of the GREEKS

The most remarkable developments of Greek mathematics can be summarized as (i) the development of our ideas about the properties of numbers and the operations one can do with them; (ii) the development of what we now call "Euclidean geometry" to a very high level, and the connection between geometric concepts and numbers; and (iii) the development of what is now called the 'Axiomatic method". Thus were laid much of the foundations of the modern discipline of mathematics (along with later contributions from Islamic and renaissance mathematicians, notably to the development of algebra). We will discuss each of these developments in turn.

2(A) Numbers, Series, and Infinitesimals

The development of numbers went to some extent in parallel with the development of astronomy, and the roots of these are buried in prehistoric times. This is hardly surprising- it is known that a number of birds and mammals are capable of counting, and it is obvious that prehistoric man could do this. What was crucial was the development of a written notation for keeping track of numbers (ie., accounting), and for manipulating them. These skills were already possessed by the Babylonians and Assyrians, but it is perhaps surprising how much depended on having an adequate notation to describe numbers and their operations. From the slides one can see how this notation developed, but the history is long and tortuous. It is worthwhile noting that the refinement of mathematical notation goes on today, and has often been associated with key advances in the subject. For more on this subject, see some of the references (in "supplementary material").

The deep interest of the Greeks in the properties of numbers apparently began with the Pythagorean school, and later with Democritus. These philosophers were interested in the extraordinary variety of properties that numbers, and collections of them, can display. Amongst other things this led to an interest in irrational numbers, and how to approximate them, but gradually a very sophisticated understanding of number theory was built up. By the time of Plato, extensive results were known- important figures being Theodorus, Theatetus, and Eudoxus. however things really got going with the later Alexandrian schools. The advances eventually made by the Greeks in our understanding of number theory (still one of the most difficult and subtle branches of mathematics) were staggering. This work was driven purely by curiosity but led to discoveries of enormous importance. The most notable figures in all of this were Pythagoras, Archimedes, Euclid, and, much later, Diophantus, who invented what is now a whole field in modern mathematics (the field of 'Diophantine equations'). Some of this work is of considerable interest today, and it was essential to the revival of mathematics in Europe during the renaissance (Thus Fermat's famous "last theorem" of 1637, one of the most famous theorems in mathematics, and only very recently proved by Wiles, was found inscribed in Fermat's copy of the book *Arithmetica*, by Diophantus; this book was apparently carried everywhere by Fermat).

I what follows we are going to look at a very small part of all of this. We will examine (i) the understanding of infinitesimals and infinite series possessed by the Greeks, and (ii) the discussion by the Greeks of rational and irrational numbers. In the background to all of this discussion is the key fact of the relationship between numbers and geometry, which is a whole subject in itself.

(ii) SERIES, APPROXIMATIONS, and the BEGINNINGS of CALCULUS: The example given in the slides (Democritus's method of finding the area of a triangle, which he also generalized to find the volume of a cone) is useful because simple. More complicated examples of note were the problem of 'squaring the circle (ie., finding out π), and of 'doubling the cube' (ie., finding out ${}^{3}\sqrt{2}$). However it is important to stress that this sort of thinking led to 2 very important developments:

(a) The idea that successive approximation to some quantity could give a kind of "limiting operation". Even if the approximation never terminated, if successive terms ever more closely approximated the correct answer, then the approximation was useful. In the same way one could construct an infinite series which summed to a finite value (eg., the series $S = 1 + 1/2 + 1/4 + ...1/2^n + ...$, which in the limit of an infinite number of terms gives the simple answer $S \rightarrow 2$). The use of such series and approximations became particularly important to the Greeks once the existence of irrational numbers was understood, (a fraction does not need such manoeuvres for its evaluation, whereas to give an approximation for an irrational number, one needs to manipulate series of this kind). This is not to say that the Greeks accepted infinite series equably - the paradoxes of Zeno discussed in the slides show that they worried a lot about them - but the work of Archimedes showed them how to deal with them.

(b) An important step was taken when attempts were made to evaluate the areas of *curves* (as opposed to simple straight line figures). This is a much harder problem, but in the slides it is shown how it was done for the circle, by dividing it into successively smaller triangles (leading to the approximate evaluation of π). In the hands of Archimedes, these techniques of dividing areas and volumes into increments, making these successively smaller, and then calculating results for areas, volumes, and centre of gravities, were developed into a fine art, with strict proofs for all the results. In this sense Archimedes invented the integral calculus, although in a less streamlined form than the later renaissance re-invention.

(c) Although the Greeks were not to acquire a full understanding of this at the time, these methods of infinitesimals would later provide the basis for a full understanding of the differences between rational and irrational numbers. Rational numbers are of course defined by fractions, but this does not work with irrational numbers - and the only way of defining an irrational number like π was to find some series that would, if continued indefinitely, tend eventually to π. What this meant was that by continuing to evaluate higher and higher approximations to the sum of the series, one would approach ever closer to the desired answer. In this sense a series could be used to define an irrational number.

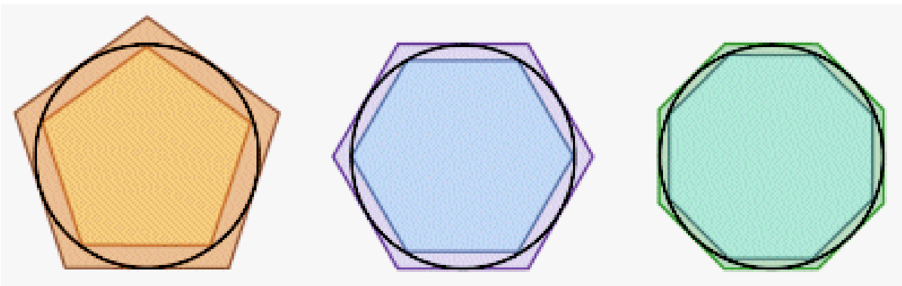

FIG. 1: The method used by Archimedes to find the area of a circle (and thence to find π). Polygons made from triangles are inscribed inside and outside the circle - as the number N of triangles increases, the total area A_N^+ of the outer polygon approaches the area A_N^- of the inner one, and the area of the circle is bracketed between the two.

Let us see how all this works by looking at the number π as an example. This number can be defined in various ways - here we simply define it as the ratio of the circumference of a circle to its diameter. Now it was already suspected by the time of Archimedes that it was irrational (a proof had to wait for another 2000 years). The technique of Archimedes for finding an approximate value for it consisted in 'bracketing' the circle of unit diameter (and hence circumference of π) between two polygons, one just outside the circle and one just inside (and each touching it). This is shown in the slides, and in Fig. 1 here. The basic idea is that if the radius of the circle is $r = 1$, then it area is $A = \pi r^2 = \pi$. Since the area of the circle is somewhere between the area of the outer larger polygon and the area of the smaller inner one, then by finding the areas of these 2 polygons, we can estimate the value of π.

Now it is not too difficult to evaluate the areas of the two polygons (we simply have evaluate the areas of the triangles making up the polygon). The key to this method is that their areas rapidly approach each other as one increases the number of sides of each polygon. Suppose we call the area of the larger polygon of N sides by A_N^+ and the area of the smaller polygon of N sides by A_N^-. We see from the figure that as N gets bigger, the difference $\delta A(N) = A_N^+ - A_N^-$ gets smaller and smaller very quickly (this difference is of course just the area enclosed between the two polygons). Thus we close in very quickly on π from both above and below - it is trapped between the two - and as N gets bigger and bigger, we rapidly approach π from both sides. Nowadays we would say that

$$\lim_{N\to\infty} \delta A(N) \rightarrow 0 \, ; \qquad \lim_{N\to\infty} (A_N^+ - \pi) = \lim_{N\to\infty} (A_N^+ - \pi) \; = \; 0 \tag{0.1}$$

which is another way of saying the same thing.

Now Archimedes evaluated the areas A_N^+ and A_N^- for $N = 96$, ie., for a polygon of 96 sides, and thereby found that $223/71 > \pi > 220/70$. This was a very good approximation for his day. To see how good it is, note that a decimal expansion of these two numbers gives $A_{96}^- = 221/71 = 3.14084507...$ and $A_{96}^+ = 220/70 = 3.14285714....$, so that the difference between them is $\delta A(96) = 0.00201207....$, ie., roughly $1/500$. Nowadays it reasonably easy for a computer, using a better series expansion than Archimedes, to get a value for π in a decimal expansion out to many *trillions* of decimal places, and with the right expansion programmed into your laptop, you can get it out to several billion decimal places. The first few terms in this decimal expansion are

$$\pi = 3.1415926535897932384626433832795028841971693993751 0....$$

and it unlikely you will ever need more than the first few terms of this - for a circle the size of the visible universe, the error in the ratio taken from the above result would amount to a distance hundreds of trillions of times smaller than the atomic nucleus.

But what does this decimal expansion really mean? In fact, like any decimal expansion, it is just the first few terms of an infinite series! We can write it as

$$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10,000} + \frac{9}{100,000} + ... \tag{0.2}$$

In other words, this is exactly the kind of series that Archimedes was trying to find. And by saying that π is irrational, we mean that this series will never end, and that it will never show any kind of recurring pattern. For *if it did*, then this would mean that we had found a finite series for π, which would then be writable as a fraction.

Notice that from this picture of an irrational as a decimal expansion with no pattern or end, it is easy to see why the irrationals outnumber the rationals on the line (even though they are both infinite in number). Imagine we had a computer which randomly generated numbers x between 0 and 1 on the line, by generating arbitrary sequences of numbers in a decimal expansion. To produce a rational fraction, this expansion would either have to look like

$$x = 0.4231400000000000000..... \quad = \frac{42314}{100,000} \tag{0.3}$$

or like

$$x = 0.142857142857142857142857142857142857142857...... = \frac{1}{7} \tag{0.4}$$

in which the sequence 142857 repeats indefinitely. Now it is obvious that to get either of these is infinitely improbable - the computer will continue to generate higher and higher terms in the decimal expansion, and we are asking that no matter high it goes, it never breaks the pattern (either of repeated zeroes, or of repeated sequences of numbers). This is clearly going to happen with zero probability, even though there are an infinite number of such expansions between 0 and 1. Thus we see that the number or irrational numbers is infinitely greater than the number of rationals in any interval, even though the number of both is infinite. And if you think this is confusing, imagine how it was for the Greeks, who did not have the advantage of decimal expansions (the zero was introduced much later, by Indian and Arab mathematicians).

Finally, we note that Archimedes was able to extend his use of series to compute many other areas. Another example is shown in the slides - he calculated the area of a section of a parabola, by first showing by geometric reasoning that the area of triangles inscribed into the regions between a given triangle and a parabola had to have an area of $1/8$ of the original triangle. Since the number of such triangles is two, this meant that the total area of the section of the parabola had to be

$$A_P = A_T \left[1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ... \right] \tag{0.5}$$

where A_P is the area of the parabolic section and A_T the area of the first triangle. This sum he easily found to be $A_P = 4A_T/3$, in other words, the answer was a rational fraction, and he found the exact result.

It is unfortunate that because of the vicissitudes of history at that time, and later on, most of Archimedes's work of this kind was ignored or not understood by subsequent workers. It was not until nearly 2,000 years later that it was finally extended by Newton, Leibniz, etc., to form part of the scientific revolution at that time.

2(B) Greek Geometry & the Axiomatic Method

The axiomatic method was an attempt to develop in a logical way the essential elements of mathematics as then understood, starting from a basic set of propositions. These basic propositions are called '**axioms**'. Although the origins of this method are shrouded in mystery, it was clearly a response to the uncertainty surrounding both the paradoxes in number theory, and the relationship of numbers to geometry.

(i) The AXIOMATIC METHOD: An axiom is a single proposition which is assumed, without proof or any other antecedent formal demonstration. All theorems/propositions of an axiomatic system are then derived from a few axioms using the rules of the system (the 'rules of inference'). Note that there is no requirement for the axioms to be 'true' or self-evident - they are simply the starting point for reasoning in some given logical system.

A simple way to think about this is to imagine that we have some set of symbols (an 'alphabet'), which can be strung together in 'strings' (think of these as 'phrases'), according to certain rules. Then other rules of manipulation/generation allow us to derive/generate other strings from a given string. No meaning is assigned to these strings of symbols - they are just symbols. To start this off we need one or more initial strings - the 'axioms'. Then, using the rules of manipulation, we can generate a large number of other strings, produced by applying the rules over and over again.

How do we mow assign *meaning* to these strings? In logic this is done by assigning some correspondence between the strings and a set of objects in some other (usually larger and richer) system. If the objects in this second system actually exist in the 'real world' of experience, then we can assign meaning to them in this way. More usually, the second system is also a formal system, and we may have to go further, to some higher system again, perhaps several times, before we can eventually map to objects in the physical world. These higher formal systems are called 'metasystems', and by mapping to them from the original system, we say we have 'interpreted' the original system.

Perhaps the easiest way to think of all this in a modern context is to imagine that the original formal system is a set of symbols and strings that are readable by a computer, which has in its software the rules of manipulation and the axioms. Given an input of some string or strings) it can then generate more strings. This is what your laptop does if, eg., you ask it to do a simple arithmetical calculation. You type in the question, which is then converted to a set of instructions (input) in the computer symbol language. It then quickly carries out the transformations, using the software, to generate the answer - and this is then translated back to numbers that you read on the screen.

Another way to think about all of this, if you like to think visually, is to imagine attaching each sentence/propostion/string in our logical system to a "node" in a graph (think of each node as being like a 'base' in baseball or softball). Then the initial set of strings, or axioms, is just the set of statements living on the starting set of nodes - we can think of each starting node as denoting one of the axioms. After that, we have all the different propositions or strings that can be generated by the rules if we start from the initial axioms - these are represented by a large number of other nodes (typically infinite in number unless the logical system is simple). To go from the axioms to one particular node, we have to apply the rules of the logical system, which tells us what moves we are allowed to make between nodes.

Let's look at some examples to see what is going on here.

Example 1: The AXIOMATIC COLOUR GAME

To see an example of the node picture just discussed, consider Fig.**?** . Here the initial axioms ar just 3 different colours, and the rules of inference specify that we are allowed to make other colours with these - any node (ie., string) which has this colour can be connected to from the axioms. We can continue this process to generate a whole variety of colours, in principle colouring an infinite number of different nodes. If you like, think of a computer program which is supposed to be running a paint factory, and which is controlling machinery which mixes different paints to make other paints of different colours. To keep it simple, we are only allowed to mix 2 paints at once.

Both the picture shown in Fig.**?** , and the idea of a paint factory just described, are examples of 'metasystems' in the real world, which we use to 'interpret' the original logical system. If we had to program a computer to control the paint factory, we would not be allowed to use names like 'green', or talk about paints. We would have to give a set of coded instructions, in which paint pots were labeled by symbols. And once this is done, you see that the paints themselves are irrelevant to the structure of the system - all that matters is the way in which we are allowed to manipulate the strings of symbols (which are coded commands) and what are the initial strings.

Another very useful thing comes out of the paint pot interpretation, or by looking at Fig.**?** . Notice that there are paints that we can never make using the procedure specified above. White, black, and thus grey are all inaccessible - they cannot be produced by mixing colours starting from the primary colours. Thus they are 'off the grid'; these are colours which are beyond the capability of a system which starts off with only these 3 primary colours and then mixes them. In the same way, we can imagine starting with a set of axioms, written in terms of symbols (ie., 'strings' of symbols) and rules for manipulating the strings to make other strings, which allow us to produce an infinite set of different strings - but that it is possible to write strings that are perfectly legitimate strings in this logical system, but which can never be reached starting from the initial strings. These would be 'outliers' strings, off the grid. To put it another way, we can imagine that there are perfectly legitmate 'sentences' or 'phrases' in this 'language', which can never be generated starting from the axioms and rules of the language.

If a logical axiomatic system has 'outlying strings' like this, we say that this axiomatic system is *incomplete* - that there are strings that the axioms cannot generate. Of course one can always imagine adding more axioms to make it complete. In the case of the colour game it is pretty obvious how one might do this. We could for example, add black

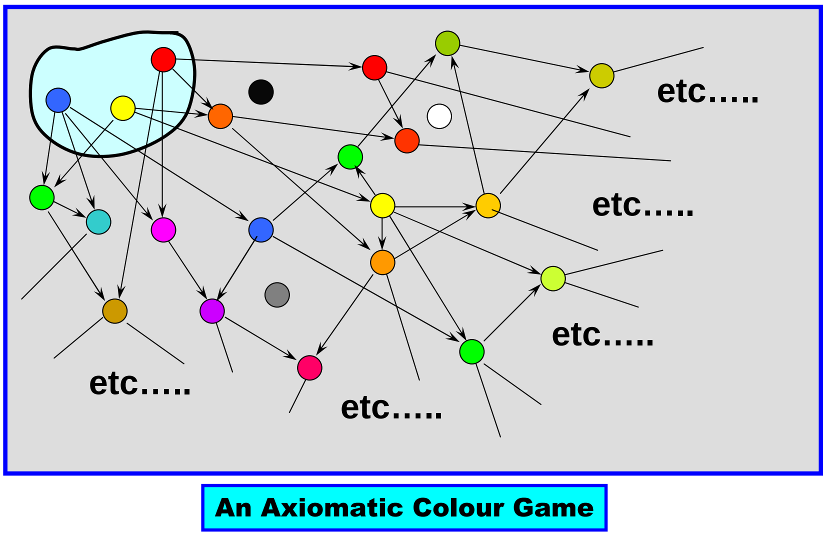

FIG. 2: In this 'colour game', we start from the nodes (the 'axioms', which carry the primary colours red, yellow, and blue, respectively. We are then allowed to mix the colours from a maximum of two nodes, to connect to another node having the same colour as this mixture. Thus if we take the red and yellow axiom nodes, we can connect to an orange node; blue and yellow nodes can connect to a green node; and so on. This procedure can generate connections to an infinite number of nodes of different colours. Notice, however, that we cannot connect to nodes that are white, black, or grey - these colours cannot be made by mixing any of the primary colours, or colours made from them.

and white nodes to the axioms, to give us 5 axioms. The system would indeed then be complete.

Example 2: RULES of CHESS: an AXIOMATIC SYSTEM

We are all familiar with the game of chess. Let's first see how we would write down the rules for this game in a way which refers explicitly to the physical objects in the game (the pieces and the board). Thus we begin by discussing an *interpretation* of the game of chess. This not the logical system of chess, and so it is not a description which would be useful to a computer. But it is useful to us. Thus, let us define chess as follows:

1. Chess is a game played between two players.

2. The players play on an 8 by 8 board, with 64 squares, in 2 alternating different colors (eg., red and yellow). Players are named also by different colours (eg., black and white), and so are their corresponding pieces.

3. Each player has 16 chess pieces, as follows: eight Pawns, a King, a Queen, two Bishops, two Knights, and two Rooks. At the beginning of the game they are placed in a certain starting position which we can take to be the initial 'string' of this formal system.

4. Players than make moves, by moving a single piece according to the rules of movement. Players move in turn; White makes the first move.

5. Each piece has its own unique move, specified by rules unique to that piece; and there is also a rule for how they can capture (and thereby remove) any opponents piece. Thus, eg., the pawns move forward one step at a time, except for their first move where they can choose to forward two squares. They can capture (ie., remove) any one of the opponents pieces by moving one square forward and diagonally onto the square occupied by the opponents piece. Other rules govern the movements of each of the other pieces.

6. Whichever side captures the opposite players King wins the game.

Now this is not a complete description of the rules - it is simply enough to give you a taste for what is going on. We see that all this is perfectly understandable to a person, as a set of instructions, provided the person has a board and pieces available, and can identify the pieces. These latter pieces are the *physical manifestation* of the game. But, as we also see, such a description is useless to a computer, or as a logical formal system. Let's try and do a little better by going to a more formal 'coded' description, of the kind used in newspapers to describe a chess game. This works as follows:

1. Each square of the chessboard is identified by an "xy" coordinate pair consisting of a letter and a number. The x-coordinates are labeled a through h, and the y-coordinates are numbered 1 to 8. Thus, eg., the white king starts the game on square e1, the black rooks on h1 and h8.

2. Each piece is symbolized by an uppercase letter, viz., K for king, Q for queen, R for rook, B for bishop, and N for knight, etc. Pawns are sometimes indicated by P, sometimes not indicated by a letter, but rather by the absence of any letter (one need not distinguish between pawns for moves, since only one pawn can move to a given square.

3. Each move in the game is denoted by the relevant piece's name, plus the coordinate of the square it moves to. Thus Ng3 moves a knight to g3. When a piece makes a capture, an x is inserted between the piece's letter and the final square. Thus Qxf7 means Queen captures the piece on f7.

4. A sequence of moves then looks as follows:

	White	Black
1.	Pc3	Pc4
2.	Nf4	Nc6
3.	Bd7	Pa4
	etc.	

Again, this is not an exhaustive description of the rules.

Now we see that this scheme is a lot better. If we eliminate all discussion of 'squares', and just talk about the coordinate codes, the symbols, etc., and give further details on the rules of manipulation of the pieces, then we could program a computer to actually generate an allowed sequence of moves. In reality, computer programs of this kind use some standard computer language, and so to really get a computer to play chess, we have to convert everything to this language, for which further work is needed. A well-known example is 'Gnu Chess' (which you can look up on the internet).

Hopefully this gives you a good idea of what we are talking about. We see that the original formal chess system is just the set of instructions and rules fed into the computer. The 'theorems' or 'propositions' that are then generated are just the sequences of moves, as written in code. We see that a finite but very large set of theorems can be generated - the set of all possible games of chess. It is finite because the rules actually do not allow for a situation where moves are repeated in the end game - this automatically leads to a draw. Nobody knows the total number of possible games, and it will be very large indeed (one can roughly estimate a number of order 10^{200}, which is enormously larger than the total number of atoms in the universe, of order 10^{80}).

Then we can make the mapping or correspondence between the formal logical chess system, and the physical objects that embody it - the chess board and pieces. This can pass through an intermediate stage, in which the game is represented by drawings on paper, showing the situation after each move. It is clear that this visual representation is also a formal representation of a kind, much more complicated than the original formal rules, but which 'contains' these rules in its own workings - it is 'metasystem'. It is of course completely open to anyone to speculate that even the physical embodiment of the game, in terms of the chess board and pieces, is also nothing but some formal representation embedded in an even more complex 'higher reality'. This of course leads to the idea that the physical world is nothing but a kind of 'program' in some awesome "supercomputer". The film 'The Matrix' is a speculation (containing many inconsistencies) of this kind.

Finally, let us note that the chess game, just like the colour game, is also incomplete. We can very easily set up situations on the chess board that are completely inaccessible from a starting position, if we apply the existing rules of chess. For example, any position of the pieces in which the 2 kings happen to be in adjacent squares is impossible (in fact it is in contradiction to the rules). Or any situation in which some of, eg., the white pawns are behind their lines, in the rearmost file as seen from white's side, is impossible (pawns cannot go backwards). And so on.

Of course we could, by adding new rules, try and fix this, and make all positions accessible. But here an interesting question arises - can we do so without making the game internally inconsistent, in such a way that illegal moves are generated by the new rules? This is an interesting problem you might like to think about.

(ii) The AXIOMATIZATION of EUCLIDEAN GEOMETRY: The idea in the axiomatisation of geometry is that one specifies the relations between a set of primitive entities by a set of geometrical axioms (in the case of geometry these are points and lines), along with a set of rules for manipulating them. At the same time as fixing the rules of operation with the primitive entities, one can also provide an *interpretation* of the system, again by specifying how to relate the objects/entities to objects in the real world- these being geometrical figures, in the case of geometry. However, what was crucial to the whole exercise, as started by Euclid, is that this latter step is *not necessary.* In other words, one can deal with propositions about points and lines, derived using the axiomatic method from a primitive set of propositions (the axioms), without having the slightest idea what these words and propositions might refer to.

The imaginative leap to such an abstraction was quite prodigious. In the 21st century we can think again about it as follows- imagine that the axioms are provided to a computer, in the form of a set of instructions about what operations are allowed on a set of primitive identities which the computer can call 'birds' and 'bees' (but which we instead like to call 'points' and 'lines'). The crucial (and perhaps non-intuitive) thing is that any 'meaning' of the terms "point" or "line", and all statements about them, is entirely acquired via the axioms. As noted above, the 'meaning' nowadays is usually taken to refer to an *interpretation* of the basic entities and of the theorems in the logical system, by connecting them to objects either in some larger logical system, or to objects in the real world. But all we really have in the original logical system is the theorems - this is just the set of all propositions in the logical system that can be derived from the axioms using the rules of inference. These propositions are 'true' if they can be derived from the axioms, and false if their contrary can be derived. Thus "true" here just means 'derivable from the axioms'.

If we then wish to interpret the objects in the formal system as objects in the real world (eg., as lines, points, etc., in 3-d space) then we are at liberty to do so. However, in the modern view, the only way of deciding if this interpretation is a correct one (in the sense that true propositions about objects in the logical system are also true of the corresponding objects in the real world) is by experimentally checking in the real world. If this correspondence is valid, then we can, if we wish to, talk about the meaning of the propositions in the logical system as though they really corresponded to statements about the real world.

As a measure of what an enormous step this was, we can look at Euclid's original formulation of geometry, and then look at what happened to one of the features discovered by Euclid that perplexed him. This is the story of the famous "5th axiom", or the "axiom of parallels". But let us begin with the Euclidean formulation. Euclid did not formulate things in exactly the way that we would do now, in terms of a set of symbols, rules of manipulation, and axioms (this would have been too much to ask!). Instead he set up a set of definitions, postulates, and 'common notions', as follows:

DEFINITIONS, POSTULATES, and AXIOMS in EUCLID'S "ELEMENTS"

Definitions

Definition 1. A point is that which has no part.

Definition 2. A line is breadthless length.

Definition 3. The ends of a line are points.

Definition 4. A straight line is a line which lies evenly with the points on itself.

Definition 5. A surface is that which has length and breadth only.

Definition 6. The edges of a surface are lines.

Definition 7. A plane surface is a surface which lies evenly with the straight lines on itself.

Definition 8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Definition 9. And when the lines containing the angle are straight, the angle is called rectilinear.

Definition 10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Definition 11. An obtuse angle is an angle greater than a right angle.

Definition 12. An acute angle is an angle less than a right angle.

Definition 13. A boundary is that which is an extremity of anything.

Definition 14. A figure is that which is contained by any boundary or boundaries.

Definition 15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 16. And the point is called the center of the circle.

Definition 17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Definition 18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Definition 19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Definition 20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definition 21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Definition 22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Definition 23 Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates
Postulate 1. One may draw a straight line from any point to any point.
Postulate 2. One may extend a finite straight line continuously in a straight line.
Postulate 3. One may describe a circle with any center and radius.
Postulate 4. All right angles equal one another.
Postulate 5. It is the case that, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, intersect on the side where the angles are less than the two right angles.

Common Notions
Common notion 1. Things which equal the same thing also equal one another.
Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.

Now let us see what is going on here. The big difference with a modern formulation is that nowadays we attempt to strip away all meaning or 'semantic' content from the initial rules, etc.; they are just symbols and rules of manipulation. This means that in a modern formulation one must have 'primitive' notions that are not explicitly defined at all - in fact they can only acquire meaning through some interpretation. It is only with the rise of formal logic in the last 150 years that the need for this way of doing things has been appreciated, partly under the stimulus of the discovery of non-Euclidean geometry. It is no exaggeration to say that without this axiomatic way of doing things, neither modern mathematics nor modern computing (which relies on formal languages, useable by a machine) could have ever started.

To Euclid it was not at all clear that one should do this. Nevertheless he was very concenred to set up a logical system in which one could deduce, with absolute certainty, the truth (or falsehood) of propositions of geometry and arithmetic. We can see that his postulates and 'common notions'; are very roughly equivalent to our modern axioms.

Now the postulate of key interest here is the 5th one (sometimes also called 'Euclid's 5th axiom). Let us rewrite it in an equivalent form, viz.,

"Given a straight line, and a point P not on the line, there is one and only one line passing through this point P which is parallel to the first line."

Yet another way to put this is that there is only one line through the point P which will not cross the first line, if they are both extended to infinity. What bugged Euclid, and hundreds of mathematicians for 2000 years after him, is that this axiom seemed on the face of it to be unnecessary. The proposition seemed so self-evidently true of real figures that it was hard to see how it didn't follow from the other postulates and common notions. And yet Euclid took a rigorous approach and found that he could not prove it from these other axioms- it was apparently necessary to add it as an independent axiom. Thus arose the famous problem- how to either (i) find a proof that the 5th axiom did follow from the others, or (ii) shwo that it did not, and was thus logically independent of the 4 others, and was therefore necessary to make Euclidean geometry a complete system.

(iii) AXIOMATIC SYSTEMS: MODERN DEVELOPMENTS : It is very likely not obvious to you why the development of the axiomatic method was so incredibly important in the history of human thought. In fact it took 2000 years for its enormous power to begin to be realized. We shall look again at this later on, when we get to modern physics, but here is a very brief foretaste of what it is to come.

(a) NON-EUCLIDEAN GEOMETRY: As noted above, the ancients left mathematicians with the great puzzle of how to prove that Euclid's 5th axiom either was, or was not, independent of the first four axioms. Many such 'proofs' were devised over the following 2000 yrs, but they were all incorrect.

Finally, in 1829, the shattering resolution of the problem was published by Lobachevski, and independently in 1832 by Bolyai (although it seems Gauss had already made the discovery in 1824). These mathematicians took the bold step of *denying* the 5th postulate, ie., assuming that the 5th axiom *was* independent of the first four, and that moreover it was *FALSE*. That it took 2000 years to make this step can be explained by noting that the 5th axiom really does seem to be utterly self-evident to most people, and hardly possible to deny. By denying the 5th axiom, these writers showed that one could get not just one, but an infinite variety of other "non-Euclidean" geometries, simply by introducing other axioms in its place. In other words, the 5th postulate was not only independent of the first four; it was *necessary* to use it in order to define Euclidean geometry at all (along with the other notions and definitions).

Finally, in the 20th century, Einstein capped off all this mathematical development by showing that the universe actually was described by non-Euclidean geometry! We will get to this later, in discussing relativity. But the key point to be emphasized here is that Euclid, in developing the axiomatic method, and sticking to it, had opened the door to possibilities which could not have been arrived at, or even dreamed of, if one had relied solely from observations of the world, or on intuition based on experience. Thus his work was of enormous philosophical importance, as well as having shattering scientific consequences.

At the time Euclid did this work, the implications of it were too far-reaching for the Greek world. In fact the work of Pappus 600 years later was still further developing Euclidean geometry; and Kepler, Descartes, Newton, Laplace, Maxwell, etc., working in the 16th-19th centuries, never questioned Euclidean geometry as a description of the real world. Perhaps the most profound thinking on the philosophical aspects of this question came from Immanuel Kant, in the late 18th century - he realized that our intuitive understanding of both space and time had to be based partly on our own mental and sensory capacities, and that there were very important limitations inherent to these mental/sensory faculties. But it certainly never occurred to Kant to doubt that spatial geometry was Euclidean, or to even imagine any other possibility. If there is a clear lesson to this, it is that the axiomatic method opened up not just a new realm of mathematical thinking, in this case about geometry, but also provided the door to a new view of the universe - which Einstein finally opened and stepped through. And there is also a clear moral - that our deepest intuitions about ourselves and the universe, ideas that seem utterly self-evident and beyond question, may just be completely wrong.

(b) GODEL'S THEOREMS: Even more astonishing than the result of non-Euclidean geometry was the purely logical result obtained by Kurt Gödel in 1931 - certainly the most important result in 20th century mathematics, and potentially the most far-reaching result in the whole history of mathematics (although we may have to wait another 2000 years to find out!). Gödel's result is rather technical at first glance, and for a non-mathematician the proof is not easy to follow. However we can state the results fairly simply as follows:

(i) Any sufficiently complicated formal axiomatic system that is complete must be inconsistent.

(ii) Any sufficiently complicated formal axiomatic system that is consistent must be incomplete.

Now by 'sufficiently complicated' we actually mean pretty simple - for example, Gödel showed that ordinary arithmetic is certainly complicated enough (and try thinking again about the chess system discussed above in this context!). So, what Gödel was really saying was that pretty much any non-trivial logical system was going to be subject to his theorems.

We can begin to see what Gödel's theorems mean if we go back to the Colour game. There we saw that the system was incomplete - there were strings or 'theorems', or 'propositions' that can be stated or articulated in this system that are 'off the grid', ie., inaccessible from the axioms. We saw the same with the chess game. Thus the result that a system can be incomplete, in this way, is no surprise. However what does it mean to say of a system that it is *consistent* (or inconsistent)? Basically it means that it contain no contradictions. For example, in arithmetic we would definitely have a contradiction if we could prove, starting from the axioms, two different results that contradicted each other (if, eg., we could prove some result and also prove its contrary).

To see how this would work in our 'node' picture of a logical system, we would have to associate 'strings' to each node, that were more complicated than simple colours - they would have be statements or propositions of some sort. And a contradiction would then arise if a proposition P_1, at some node n_1, contradicted another proposition P_2 at another node n_2, and if both of these nodes could be reached starting from the axioms. For this would then imply that we could derive or 'prove' P_2 starting from P_1, ie., that the truth of P_1 implied its falsity.

It is actually very easy to make up contradictions of this kind. Some of them were known to the ancient Greeks - for example, the paradox of the Cretan who stated that "all Cretans lie all the time". We see that if this particular Cretan is telling the truth, by asserting that all Cretans are always liars, we get a contradiction, because he is also Cretan, and so is an example of one Cretan who does not always lie. In other words, if he is telling the truth then he is a liar. Likewise, if the Cretan is in fact a liar, then he has just uttered a true statement, implying he is not a liar. Thus we get contradictions both ways. For 2000 years it was assumed that one could argue either that such statements were meaningless (which is hard to defend, since they clearly mean something), or that any proper logical system would simply make it impossible to make such statements (ie., a proper logical system would be a consistent system). It was not until the early 20th century, primarily in the work of Bertrand Russell, that the full importance of such paradoxes began to be realized, when mathematicians tried to derive all of mathematics from logic (in an attempt to derive 'true' results in mathematics, and to make it possible to base mathematical knowledge on completely logical foundations). It was then found to be extraordinarily difficult to eliminate contradictions - indeed, all efforts seemed to be either unworkable or extremely artificial.

Even in spite of this, the result of Gödel came as a huge shock to most mathematicians and logicians - just as with Euclidean geometry, the idea of consistency in any correctly constructed logical system seemed completely self-evident. We now understand much better what had been partially grasped by Russell and others, that simple paradoxes such as the Cretan liar paradox exemplify a much deeper problem in logic, which we can call the problem of 'self-reference'. Once it is possible to make statements (ie., compose strings or propositions) in some logical system which in some way make reference to properties of the logical system itself, then contradictions can arise. Three further examples of 'self-referential' paradoxes will help to make this point:

(i) The *Barber's paradox*: Consider a barber in a town who says that he will shave all those people in the town who do not shave themselves. Question: who will shave the barber?

(ii) An "*indexing paradox*": Consider a system, like Google, that makes lists of items according to some search criterion, and then create webpage with these lists. For example, we can ask it to list all websites describing movie actors. This list will have many millions of entries. Or we can ask it to list all articles about Moose Jaw - this will not have so many entries. Note that neither of these 2 lists will refer to itself, ie., it will not contain itself as an entry. However, let's ask it to list all articles about lists. The resulting webpage will probably have quite a few entries - and notice that it will also list itself, since it is also a list. To create a paradox, we now ask the system to list all lists that do not list themselves. The paradox: if the list does not list itself, then it is not a list. If however it does list itself, then then it is listing itself as a list that does not list itself. Presumably Google does not attempt to create list webpages of this kind!

(iii) *Russell's paradox*: This is an elaboration of the last one. We can group objects of any kind into sets. Examples: the set of all cars; or the set of all terrestrial objects. Nether of these sets contains itself as a member (eg., the set of cars is not a car). However consider the set of all non-terrestrial objects - this is also a non-terrestrial object and so is contained in (ie., is a member of) itself. Now consider the set of ALL sets that are not members of themselves. Question: is it a member of itself? We see that if it is, it isn't; and if it isn't, it is. We are faced with the same kind of contradiction as before.

From examples like this we see that self-reference is a kind of Achilles heel for any kind of logical system. In fact Gödel's proof relied on a very clever mapping of the entire system of arithmetical propositions onto different numbers (the Gödel numbers) which could then themselves be subject to arithmetical operations - and these operations were in effect making statements about arithmetic. In this way he then succeeded in demonstrating a contradiction. From there he was able to show that any sufficiently complex logical system (in particular, anything as complicated as arithmetic) would have similar problems. In all such systems, whenever the logical system attempts to make statements about itself, the "self-referential statements" invariably 'feed back' into the chain of logical reasoning to produce contradictions, unless (i) they are somehow forbidden (the recipe that Russell and others attempted, without real success), or else (ii) the rules and/or axioms of the system are somehow relaxed, in order to allow it to be incomplete. That the latter must happen - ie., that to eliminate contradictions the system must be incomplete - is part of Gödel's result.

Even now, 80 yrs later, we have hardly seen the full impact of Gödel's results on mathematics, let alone on the rest of science. And yet the implications for essentially all of human thinking are potentially colossal. This is just because they open up any kind of self-referential thinking to paradox and contradiction. Since the work of Gödel many people have speculated, usually in a rather imprecise way, on the possible connection between these ideas and all the difficulties we have with introspective thinking (ie., 'thinking about ourselves', or 'thinking about our thoughts'). The difficulties we have with such introspective thinking (mirrored in contemporary discussions of the meaning of 'consciousness', or of the 'Self') are of course connected with many of the oldest and most profound philosophical questions. In the same way there are speculations about what connection there might be with self-referential processes in the physical world - and in particular, with the the mysteries of quantum mechanics. Such ideas have so far led to no really clear insights, only tantalizing hints of perhaps really deep truths waiting to be discovered.

Perhaps the safest thing one can say about Gödel's results, at this time, is that they are so profound and far-reaching that we will not fully appreciate their meaning and consequences for a very long time, probably many centuries from now.