Name ____________________________
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# Physics 521 Group Theory

**Midterm Exam.**

**Please read the questions carefully. Please write in ink. This is a closed book exam. Books, notes, calculator or reference material of any kind are NOT allowed. Please write on the exam paper in the space after each question. The marks for each question are displayed. The total is 15 marks.**

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## An Application of the Dihedral Group $\mathbb{D}_4$

Consider the dihedral group $\mathbb{D}_4$ which has presentation

$$\langle c, b \mid b^2 = e, c^4 = e, (bc)^2 = e \rangle$$

1. *Find the character table of* $\mathbb{D}_4$.

**Over** $\longrightarrow$

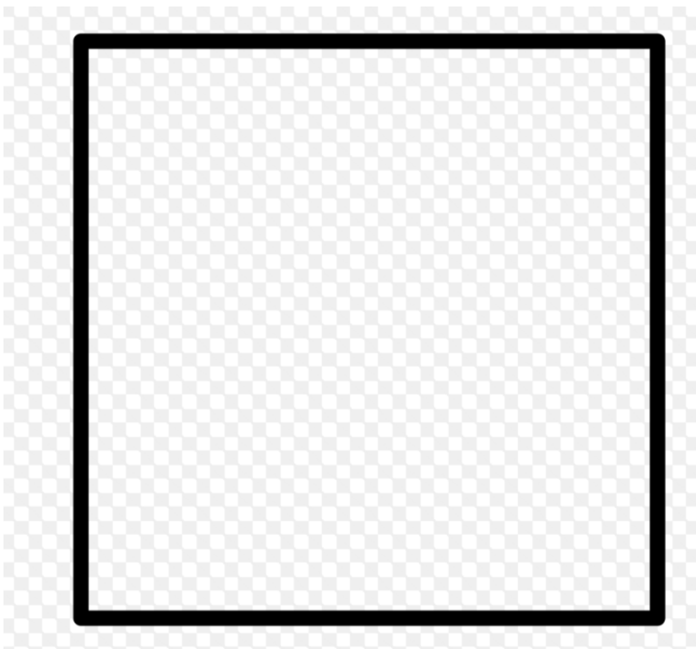

Figure 1: A square. The generators of $\mathbb{D}_4$ are $c$, a rotation by angle $\pi/2$ radians in the plane of the square and about its centre, and $b$ a flip of the square about a vertical line through its centre.

2. (3 marks) *Given an $\ell = 4$ atomic orbital, what is the crystal field splitting of its nine degenerate energy levels when the atom is immersed and is weakly interacting with an environment which has $\mathbb{D}_4$ symmetry.* Remember that the character of an SO(3) matrix corresponding to a rotation by angle $\theta$ about an axis is

$$\chi^{(\ell)}(\theta) = \frac{\sin(\ell + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}$$

**Over** $\longrightarrow$

3. (3 marks) *Take the highest dimensional irreducible representation* $D^{(\nu)}$ *of* $\mathbb{D}_4$ *and consider the direct product* $D^{(\nu)} \otimes D^{(\nu)} \otimes D^{(\nu)} \otimes D^{(\nu)}$. *This must be a reducible representation in that it can be decomposed into a direct sum of irreducible representations. Find the number of occurrences of each irreducible representation in the direct sum.*

   Character orthogonality

$$\frac{1}{n_G} \sum_{[g]} n_{[g]} \chi^{(\mu)}([g]) \chi^{(\nu)}([g])^* = \delta^{\mu\nu}$$

   might come in handy

**Over** $\longrightarrow$

4. *Given an $\ell = 4$ atomic orbital, what is the crystal field splitting of its nine degenerate energy levels when the atom is immersed and is weakly interacting with an environment which has $\mathbb{D}_4$ symmetry.* Remember that the character of an SO(3) matrix corresponding to a rotation by angle $\theta$ about an axis is

$$\chi^{(\ell)}(\theta) = \frac{\sin(\ell + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}$$

**Over** $\longrightarrow$

5. (4 marks)*Find the proper invariant subgroups $H_i$ of $\mathbb{D}_4$. Find the quotient groups*

$$\mathbb{D}_4/H_i$$

*for each of proper invariant subgroup.*

**Over** $\longrightarrow$

6. Cayley's theorem tell us that $\mathbb{D}_4$ must be a subgroup of $\mathbb{S}_{n_{\mathbb{D}_4}}$. *Identify the elements of that subgroup in cycle notation. For example* $e = (1)(2)(3),,,(n_{\mathbb{D}_4})$. *Is this subgroup an invariant subgroup of* $\mathbb{S}_{n_{\mathbb{D}_4}}$*? Is it a subgroup of the alternating group* $\mathbb{A}_{n_{\mathbb{D}_4}}$*? In the latter case, is it an invariant subgroup?*

**The end.**