Tutorial 2: The Uses of Finite Groups

1 The Symmetric Group $\mathbb{S}_4$

The symmetric group $\mathbb{S}_N$ has $N!$ elements which which are the distinct permutations, including the identity permutation, of a set of N distinguishable objects. A useful way to denote an element of the symmetric group is by its decomposition into cycles. In the following we shall consider the case of $\mathbb{S}_4$ where it is a relatively simple task to find the cycle decompositions and then all of the permutations which have a given cycle decomposition.

The set of cycle decompositions of $\mathbb{S}_4$ is equal to the number of partitions of the nunber four which we can enumerate as

$$1+1+1+1, 1+1+2, 1+3, 2+2, 4$$

The cycle structures which are associated with this partition are

$$(1)(2)(3)(4), (1)(2)(34), (1)(234), (12)(34), (1234)$$

where in each case we give an example of a permutation with the cycle structure by filling in the spaces in the cycles with integers in the sequence $1, 2, 3, 4$. The structures are, respectively, four 1-cycles, two 1-cycles and one 2-cycle, one 1-cycle and one 3-cycle, two 2-cycles, and a 4-cycle.

Then we can permute the integers in the cycle expression and all of the inequivalent permutations are distinct symmetric group elements. In doing this, cycles of the same length are considered identical objects in that (1)(2)(34) and (2)(1)(34) are the same permutation group element. Cyclic permutations within the cycles, for example (1)(234), (1)(342) and (1)(423) are also considered the same group element. However, non-cyclic permutations such as (1)(234), (1)(243) are distinct elements. We consider all inequivalent permutations and we group the inequivalent sets of cycles according to their cycle structure as follows:

- The identity permutation coincides with the four 1-cycles

$$[e] = \{(1)(2)(3)(4)\} \tag{1}$$

- The permutations containing a 2-cycle and two 1-cycles are

$$\{(12)(3)(4), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3), (34)(1)(2)\} \quad (2)$$

- The permutations containing two 2-cycles are

$$\{(12)(34), (13)(24), (14)(23)\} \quad (3)$$

- The permutations containing one 3-cycle and one 1-cycle are

$$\{(123)(4), (321, 4), (134)(2), (143)(2), (124)(3), (142)(3), (234)(1), (432)(1)\} \quad (4)$$

- The permutations containing a 4-cycle are

$$\{(1234), (1243), (1342), (1324), (1423), (1432)\} \quad (5)$$

Here we have organized the list of elements into subsets where each element of a given subset has the same cycle structure.

We recall the general fact that all of the elements of $\mathbb{S}_N$ which have the same cycle structures are members of the same conjugacy class of $\mathbb{S}_N$. This follows from the fact that conjugation of a given element with a given cycle structure simply permutes the integers in the cycles by the permutation with which one is conjugating. All conjugations by all elements of $\mathbb{S}_N$ therefore implement all such permutations which then sweep across all of elements of $\mathbb{S}_N$ which have the same cycle structure. Thus we see that all of the elements of $\mathbb{S}_N$ which have the same cycle structure are in the same conjugacy class. What is more, no conjugation can change the cycle structure. Therefore each cycle structure represents a distinct conjugacy class.

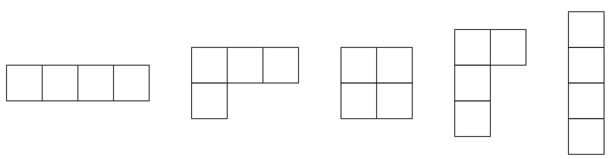
The conjugacy classes of $\mathbb{S}_4$ are thus given given by the subsets of elements of $\mathbb{S}_4$ listed in equations (1)-(5).

The 24 group elements of $\mathbb{S}_4$ are thus divided into five conjugacy classes. The number of inequivalent irreducible representations of a finite discrete group is equal to the number of its conjugacy classes. Thus we conclude that $\mathbb{S}_4$ must have five inequivalent irreducible representations. There must always be a trivial one dimensional representation, $D^{(A_1)} = 1$. This leaves four other representations which are not the trivial one. Their dimensions must satisfy the constraint that the sums of the squares of the dimensions of all of the representations must equal the order of the group,

$$24 = 1 + d_1^2 + d_2^2 + d_3^2 + d_4^2$$

> **1.** *Amongst all of the integers which are one or greater, show that the only solution of the equation* $24 = 1 + d_1^2 + d_2^2 + d_3^2 + d_4^2$ *is the one given by the set of integers* $(d_1, d_2, d_3, d_4) = (1, 2, 3, 3)$*?*
> We thus expect to find, beyond the one dimensional identity representation, another one dimensional representation, a two dimensional representation and two three dimensional representations.

A way to find the dimensions of the representations of the symmetric group is by recalling that the representations themselves are are associated with Young diagrams. The inequivalent irreducible representations of $\mathbb{S}_N$ are in one-to-one correspondence with the distinct legal Young diagrams with N boxes. The legal Young diagrams with four boxes are easy to list



(6)

and their number indeed equals five coinciding with the number of representations that we expect. In fact, we can then use the hook length formula to compute the dimensions of the representations that are associated with each of the Young diagrams. For this, we form the Young tableaux where we decorate the boxes in the Young diagrams with their hook length integers, and then compute the dimensions of the representations that are associated with them, using the hook length formula

$$\dim\,[\text{rep of } \mathbb{S}_N] \;=\; \frac{N!}{\prod\limits_{\text{hooks}} \text{hook lengths}} \tag{7}$$

The result is

$$
\begin{array}{rcl}
\begin{array}{|c|c|c|c|}\hline 4 & 3 & 2 & 1 \\ \hline\end{array} & \rightarrow & \dfrac{4!}{4\cdot 3\cdot 2\cdot 1} = 1 \\[2ex]
\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} & \rightarrow & \dfrac{4!}{4\cdot 2\cdot 1\cdot 1} = 3 \\[2ex]
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline\end{array} & \rightarrow & \dfrac{4!}{3\cdot 2\cdot 2\cdot 1} = 2 \\[2ex]
\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} & \rightarrow & \dfrac{4!}{4\cdot 1\cdot 2\cdot 1} = 3 \\[2ex]
\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array} & \rightarrow & \dfrac{4!}{1\cdot 2\cdot 3\cdot 4} = 1
\end{array}
\tag{8}
$$

This agrees with our original assessment, that there are two one, one two and two three dimensional representations.

Given the five conjugacy classes and the five irreducible representations, the character table of $\mathbb{S}_4$ is the five-by-five array

$\mathbb{S}_4$	$[e] = [(1)(2)(3)(4)]$	$6\ [(12)(3)(4)]$	$3\ [(12)(34)]$	$8\ [(123)(4)]$	$6\ [(1234)]$
A_1	1	1	1	1	1
A_2	1	-1	1	1	-1
E	2	0	2	-1	0
T_1	3	1	-1	0	-1
T_2	3	-1	-1	0	1

Recall that the symmetric groups $\mathbb{S}_N$ had a presentation where the generators are the exchanges of nearest neighbours in an ordered list of objects. The generator labeled σ_i exchanges the object in position i with the object in position $i+1$. Then the presentation is

$$
\mathbb{S}_N = \langle \sigma_1, ..., \sigma_{N-1} \mid \sigma_i^2 = e, \sigma_i\sigma_j = \sigma_j\sigma_i\ |i-j| > 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle
\tag{9}
$$

Each generator is of second order and generators acting on non-overlapping pairs of objects commute. The relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ is called the braid relation.

For the case of $\mathbb{S}_4$ such a presentation would have three generators, which we can also write in cycle notation as

$$\sigma_1 = (12)(3)(4) \ , \quad \sigma_2 = (1)(23)(4) \ , \quad \sigma_3 = (1)(2)(34)$$

with the relations $\sigma_1^2 = e$, $\sigma_2^2 = e$, $\sigma_3^2 = e$ and $\sigma_1\sigma_3 = \sigma_3\sigma_1$. In addition, there would be the braid relations

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \quad , \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$$

Group elements are products of the generators $\sigma_1, \sigma_2, \sigma_3$ which are then reduced to members of a minimal set by using the relations. It is clear from the form of the relations that a product with an even number of σ's will always be reduced to a product with an even number of σ's and a product with an odd number of σ's will always be reduced to a product with an odd number of σ's. That is, the relations conserve the number of σ's mod 2.

It is clear that the products of even numbers of generators must be a subgroup. The even products contain the identity, e (since zero is even). What is more if $\sigma_i \ldots \sigma_j$ is an even product, its inverse $\sigma_j \ldots \sigma_i$ is also even. Finally, since the relations conserve evenness or oddness, any product of two even group elements must be an even element. This is enough for the set of all even elements to be a subgroup. In fact, conjugation of an even element by any group element must be an even element. The even elements are thus a normal subgroup.

2 The Alternating Group $\mathbb{A}_4$

The normal subgroup of the symmetric group $\mathbb{S}_N$ which contains all of the even permutations is called the alternating group $\mathbb{A}_N$.

The alternating group $\mathbb{A}_4$ which is the group all even permutations of four objects. It is easy to inspect the cycle decompositions of the $\mathbb{S}_4$ elements in equations (1)-(5) and to identify which of the permutations in those cycle decompositions are even permutations and are therefore elements of $\mathbb{A}_4$. They are the twelve elements which which we list in the following four sets,

$$\begin{aligned}
&[e] = \{(1)(2)(3)(4)\} \\
&[b] = \{(12)(34), (13)(24), (14)(23)\} \\
&[c] = \{(123)(4), (134)(2), (142)(3), (243)(1)\} \\
&[cbc] = \{(132)(4), (143)(2), (124)(3), (234)(1)\}
\end{aligned} \tag{10}$$

We have retained the cycle notation for those permutations. We have also organized them into subsets which are the conjugacy classes of $\mathbb{A}_4$. The classes have inherited some of the structure of $\mathbb{S}_4$, the main difference being that the class with a 3-cycle and a 1-cycle splits into two equal sets which are distinct conjugacy classes of $\mathbb{A}_4$ whereas all elements with this cycle structure were in the same conjugacy class of $\mathbb{S}_4$.

It is easy to see by inspection that the elements within either of the two sets are obtained from each other by conjugations with even permutations only, that is $\mathbb{A}_4$ elements only. To go from one of the 3-cycle–1-cycle sets to the other requires conjugation by an odd permutation which is not an element of $\mathbb{A}_4$. but is an element of $\mathbb{S}_4$ – thus in $\mathbb{S}_4$ they were members of the same class whereas in $\mathbb{A}_4$ they are in distinct classes.

We have also suggested generators which are associated with the conjugacy classes. It is clear that each element of the $[b]$ class, say we identify $b = (12)(34)$ satisfies the relation $b^2 = e$. Similarly the $[c]$ class has elements, for example $c = (123)(4)$ which obey the relation $c^3 = e$.

2. *By explicitly multiplying the group elements, show that, when* $b = (12)(34)$ *and* $c = (123)(4)$, $cbc = (142)(3)$. *This justifies our naming the fourth class in the list of classes in (10) as* $[cbc]$ *since* cbc *occurs in that class. Why do we now know that* $(cbc)^3 = e$*?*

3. *Show that* $cb \in [c]$ *and, since each member of the class* $[c]$ *is of order three,* $(cb)^3 = e$.

4. *Show that we can choose a presentation of* $\mathbb{A}_4$ *as*

$$\mathbb{A}_4 = \langle b, c \mid c^3 = e, b^2 = e, (cb)^3 = e \rangle \tag{11}$$

Use this presentation to enumerate the group elements and demonstrate that the conjugacy classes of this group coincide with those shown in equation (10).

3 The Tetrahedral Group $\mathbb{T}$

When it is used in crystallography, $\mathbb{A}_4$ is usually called the tetrahedral group $\mathbb{T}$. It is the symmetry group of a regular tetrahedron, as depicted in figure 1 under proper rotations, that is, symmetry transformations of that object which are also elements of SO(3) that describe rotations of the tetrahedron.

It turns out that proper rotations can implement any even permutation of the four vertices of the tetrahedron thus the isomorphism between $\mathbb{T}$ and $\mathbb{A}_4$. Odd permutations, on the other hand, require a reflection as well as rotation. If reflection symmetries are allowed, the O(3) transformations which preserve the geometry of the tetrahedron coincides with $\mathbb{S}_4$ which is also called the extended tetrahedral group $\mathbb{T}_h$.

The tetrahedral group $\mathbb{T}$ is the symmetry group of the regular tetrahedron. The tetrahedron is depicted in figure 1.

This group has a presentation

$$\mathbb{T} = \langle b, c \mid c^3 = e, b^2 = e, (bc)^3 = e \rangle > \tag{12}$$

and it has 12 group elements, $n_T = 12$ which can be listed as

$$\mathbb{T} = \{e, c, c^2, b, cbc^2b, cbc, c^2bc, cbc^2, c^2bc^2\} \tag{13}$$

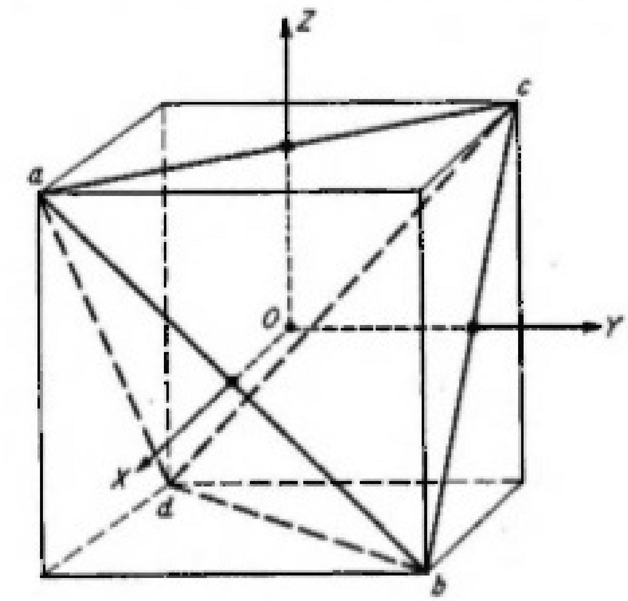


Figure 1: The regular tetrahedron is inscribed in a cube. Its four vertices are the points labeled a, b, c, d in the figure. The generator c is a rotation by angle $2\pi/3$ radians about an axis which passes through one of the vertices of the tetrahedron (one of the points a, b, c or d) and the mid-point of the face of the tetrahedron which is opposite to that vertex. There are clearly four such rotations about the four different symmetry axes. What is more, such a rotation implements a permutation of the vertices of the tetrahedron which leaves the vertex which the rotation axis passes through unchanged and which cyclically permutes the three vertices of the opposite face of the tetrahedron. They thus consist of a 3-cycle and a 1-cycle and they constitute the four elements of $[c]$. The generator b is a rotation by π radians about an axis which bisects the opposite edges of the tetrahedron. There are three pairs of opposite edges and three group elements of this sort. Such a rotation interchanges the vertices which occur at the edges of the two edges which are being rotated and thus they implement a permutation of the vertices which contains two 2-cycles. These are the transformations in the class $[b]$.

5. *Find the one dimensional representations of* $\mathbb{T}$*.*
Once you know the one dimensional representations, you know that there are four representations in total and you know that their dimensions obey the formula $12 = \sum_\nu d_\nu^2$, you should be able to deduce the dimensionality of the remaining representations and since the latter are very few you should be able to use row and column orthogonality to fill in the entire character table of $\mathbb{T}$. This keeps in mind the fact that the characters of the one dimensional representations are simply the numbers associated with group elements in the conjugacy classes.
6. *Find the character table of* $\mathbb{T}$*.*
Consider the problem which asks whether a physical system can have either a constant magnetic or a constant electric dipole moment. In an SO(3) invariant medium the answer is no. Such a moment would have to be a solution of the equation

$$D_a = R_{ab}D_b \ , \quad \forall R \in SO(3)$$

Since the matrices R lie in a nontrivial irreducible representation of SO(3), and the vector space that such a representation acts on does not have an invariant subspace, the only solution of the above equation is $D_a = 0$.
7. *Consider the same constraint, $D_a = R_{ab}D_b$, but where the matrices R must coincide with those rotations which are in the tetrahedral subgroup $\mathbb{T}$ of SO(3). Is a dipole moment allowed by symmetry in an environment with $\mathbb{T}$ symmetry?*

The conductivity tensor of a homogeneous and isotropic substance is a 3×3 matrix which must obey the equation

$$\sigma_{ab} = R_{ac}R_{bd}\sigma_{cd} \ , \quad \forall R \in SO(3)$$

It transforms in the direct product representation $D^{(\ell=1)}(R) \otimes D^{(\ell=1)}(R)$ representation of SO(3). The decomposition of that direct product representation into irreducible representations

$$D^{(\ell=1)}(R) \otimes D^{(\ell=1)}(R) = D^{(\ell=2)}(R) \oplus D^{(\ell=1)}(R) \oplus D^{(\ell=0)}(R)$$

contains the trivial $D^{(\ell=0)}(R)$ representation. This is the only representation that can have an invariant subspace – and that subspace is one-dimensional. It is the existence of this invariant subspace that allows the conductivity to be nonzero. Moreover, the conductivity tensor itself must lie in that invariant subspace. It must have the form

$$\sigma_{ab} = \sigma \delta_{ab}$$

We say that the conductivity is isotropic and it is determined by one parameter, σ.

Consider an environment where the SO(3) symmetry is reduced to the tetrahedral symmetry $\mathbb{T}$. For example, a lattice with the cubic diamond structure has this symmetry.

8. *How many parameters does the conductivity tensor have when the environment has $\mathbb{T}$ symmetry?*

9. Consider the 27-dimensional representation that is gotten by taking a direct product of three three dimensional representations $T \otimes T \otimes T$ of the group $\mathbb{T}$. This direct product must be a reducible representation of $\mathbb{T}$ and must therefor be equivalent to a direct sum of irreducible representations of $\mathbb{T}$.

$$T \otimes T \otimes T = \sum_{\nu} \oplus n_{\nu} D^{(\nu)}$$

Find the number of occurrences n_{ν} of each irreducible representation in that direct sum.

10. *Assume that the group elements in $[c]$ and $[cbc]$ are implemented by rotations by angle $2\pi/3$ radians about some axes and that the elements of $[b]$ are implemented by rotations by angle π about some axes. Find the crystal field splitting pattern for an $\ell = 2$ atomic orbital which is immersed in a crystal with tetrahedral $\mathbb{T}$ symmetry.*

11. *Find the crystal field splitting pattern for an energy level which transforms under the $T_1 \otimes T_2$ representation of $\mathbb{S}_4$ when it is immersed in an environment where the symmetry as reduced to that of its subgroup $\mathbb{A}_4 \sim \mathbb{T}$.*