Tutorial 1: The Symmetric Group $\mathbb{S}_N$

1   The symmetric group $\mathbb{S}_N$

A permutation of a set of objects in an ordered set is a rearrangement of the order of the objects in the set. The set of all such permutations of a set of N elements, with the natural composition law for consecutive permutations is a finite discrete group called the symmetric group and denoted by $\mathbb{S}_N$. The order of this group is $n_{\mathbb{S}_N} = N!$, the number of distinct permutations of a set of N elements, including the trivial identity permutation.

An element of $\mathbb{S}_N$ can be denoted by

$$\begin{bmatrix} 1 & 2 & 3 & \dots & N \\ P(1) & P(2) & P(3) & \dots & P(N) \end{bmatrix}$$

Here, $1, 2, \dots, N$ label N objects and $P(i)$ are integers which take values in $1, \dots, N$ such that, in the sequence $P(1), \dots, P(N)$, each of $1, \dots, N$ appears exactly once. The sequence $P(1), P(2), \dots P(N)$ is a re-ordering or a "permutation" of the integers between 1 and N. The instruction in the notation above tells us to replace 1 by $P(1)$, 2 by $P(2)$ and so on.

An explicit example, for the case of $N = 3$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

which carries the instructions
''replace 1 by 2"
"replace 2 by 3"
"replace 3 by 1".

The product of two permutations is the result of performing the permutations consecutively,

$$PQ = \begin{bmatrix} 1 & 2 & 3 & \dots & N \\ P(1) & P(2) & P(3) & \dots & P(N) \end{bmatrix} * \begin{bmatrix} 1 & 2 & 3 & \dots & N \\ Q(1) & Q(2) & Q(3) & \dots & Q(N) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & \dots & N \\ P(Q(1)) & P(Q(2)) & P(Q(3)) & \dots & P(Q(N)) \end{bmatrix}$$

A concrete example is

$$
\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}
$$

The middle expression above carries the instructions:
"replace 1 by 2, then replace 2 by 3, so the result is replacement of 1 by 3"
"replace 2 by 1, then replace 1 by 2, so the result is replacement of 2 by 2"
"replace 3 by 3, then replace 3 by 1, so the result is replacement of 3 by 1".

We state here without proof the fact that the product of three permutations is associative.

The set of all possible permutations has an identity. It is the permutation which leaves the sequence $\{1, 2, ..., N\}$ unchanged,

$$
e = \begin{bmatrix} 1 & 2 & 3 & \ldots & N \\ 1 & 2 & 3 & \ldots & N \end{bmatrix}
$$

Also, any permutation is invertible. The inverse is given by interchanging the two rows in the notation that we are using

$$
\begin{bmatrix} 1 & 2 & 3 & \ldots & N \\ P(1) & P(2) & P(3) & \ldots & P(N) \end{bmatrix}^{-1} = \begin{bmatrix} P(1) & P(2) & P(3) & \ldots & P(N) \\ 1 & 2 & 3 & \ldots & N \end{bmatrix}
$$

which tells us to replace $P(1)$ by 1, $P(2)$ by 2 and so on. As a concrete example, it is easy to see that

$$
\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}
$$

Indeed,

$$
\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}
$$

We simply interchange the first and second rows in the array which denotes the permutation.

1. *Show that every finite discrete group of order N is isomorphic to a subgroup of the permutation group $\mathbb{S}_N$.*

This is a statement of Cayley's Theorem.

Hint: Consider a group of order N, G=$\{g_1, g_2, \ldots g_N\}$ as an ordered set of N elements. Consider an element h of that group. Show that the ordered set $\{hg_1, hg_2, \ldots, hg_N\} = \{g_{P_h(1)}, g_{P_h(2)}, \ldots, g_{P_h(N)}\}$ is a permutation of the ordered set $\{g_1, g_2, \ldots g_N\}$. You can accomplish this by showing that each element in the set $\{hg_1, hg_2, \ldots, hg_N\}$ is a distinct element of G and since there are N distinct elements in the set and N distinct elements in G and every element of the set is an element of G, the set must be a copy of G, perhaps just in a different order.

This associates an element h of G with an element P_h of $\mathbb{S}_N$. You need to show that this association is a homomorphism, that $P_{h_1 h_2} = P_{h_1} * P_{h_2}$ for any $h_2, h_2 \in G$. Then, you also need to show that it is one-to-one, that is, if $P_{h_i} \neq P_{h_j}$ then $h_i \neq h_j$. These two facts will establish what you want to prove because:

i) The image of a homomorphism is a subgroup.

ii) The homomorphism is one-to-one and it is therefore an isomorphism onto the subgroup which is its image.

2   Cycle decomposition of a permutation

It is sometimes useful to characterize an element of the $\mathbb{S}_N$ by its decomposition into *cycles*. Finding the cycles is straightforward. For a given permutation, a cycle which is contained in it can be found as follows:

Begin with the integer 1. It is mapped to another integer $P(1)$.

Subsequently, the integer $P(1)$ is mapped to the integer $P(P(1))$.

Then $P(P(1))$ is mapped to another integer $P(P(P(1)))$ and so on.

Continue until you obtain the number that you started with, in this case 1, which, if it is after k permutations, $P^k(1) = 1$.[1] [2]

Make a ordered list of the integers that you have found,

$$\left(1, P(1), P(P(1)), \ldots P^{k-1}(1)\right)$$

---

[1] The notation $P^n(q)$ means that we have operated P n times on the integer q.

[2] The set $\{1, ..., N\}$ is finite, so this list must eventually begin to repeat integers which have already occurred on the list. That is, for some m and n, $P^m(1) = P^n(1)$ where $m > n$. Operating with the inverse of P n times tells us that $P^{(m-n)}(1) = 1$. Thus, the first repeated integer must be 1.

This ordered list is the cycle. Note that it carries the same information if the integers in the list are replaced by any cyclic permutation –

$$\left(1, P(1), P(P(1)), \ldots P^{k-1}(1)\right) = \left(P(1), P(P(1)), \ldots P^{k-1}(1), 1\right)$$

$$= \left(P(P(1)), \ldots P^{k-1}(1), 1, P(1)\right) = \ldots$$

The cycle which you have found could contain all of the integers $1, 2, ..., N$. If not, some integer, k, must be left out of it.

We could choose k and identify another cycle. This cycle must eventually return to k. Assume that his is so, after ℓ terms, that is, $P^\ell(k) = k$. Then, the cycle is

$$\left(k, P(k), \ldots P^{\ell-1}(k)\right)$$

The two cycles that we have identified cannot have any integers in common. If they had, they would be the same cycle, in contradiction to the assumption that k was left our of the first cycle.

Then, either the union of the two cycles that we have identified cover all of the integers $1, ..., N$, or not. If not, we choose another integer that is left out of the two cycles and we find its cycle.

Clearly, by our process where we find the next cycle by beginning with an integer which was left our of all cycles that were found so far, the result is that two distinct cycles cannot have any integers in common.

We continue the process until we have decomposed the entire permutation into cycles. The cycles

$$\left(1, P(1), \ldots, P^{n_1}(1)\right) \left(k, P(k), \ldots, P^{n_k}(k)\right) \ \ldots$$

form a partition of the set of integers $\{1, 2, ..., N\}$. We might guess that cycle decompositions of all of the permutations in $\mathbb{S}_N$ cover the set of all partitions of $\{1, 2, ..., N\}$ perhaps modulo the fact that inside a cycle, the integers in the cycle can be permuted cyclically. We will use this idea later.

As an example of the construction of cycles, the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix}$$

contains the cycles $1 \to 2 \to 4 \to 1$ and $3 \to 3$.

A notation that contains all of the essential information in the permutation is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix} = (124)(3)$$

An even more stream-lined notation would be to drop the (3) and assume that all integers which are not included in the cycles that are shown are members of one-cycles, that is, in the permutation, they are simply mapped onto themselves. In that notation,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix} = (124)$$

As another example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 6 & 1 & 4 & 9 & 8 & 5 & 10 & 7 & 2 \end{bmatrix} = (\ 1\ 3\ )(\ 2\ 6\ 8\ 10\ )(5\ 9\ 7)$$

Alternatively, if we know the decomposition $(1\ 3\ 2)(45)$ of an element of $\mathbb{S}_6$, for example, we can immediately associate it with the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{bmatrix}$$

We note that the identity permutation

$$e = \begin{bmatrix} 1 & 2 & 3 & \dots & N \\ 1 & 2 & 3 & \dots & N \end{bmatrix}$$

is composed entirely of cycles of length one and, in our cycle notation it would be blank, so when necessary for clarity we call it e.

The group $\mathbb{S}_3$ is isomorphic to the dihedral group $\mathbb{D}_3$ which can be presented as

$$\mathbb{S}_3 = \mathbb{D}_3 = \langle b, c \mid b^2 = e, c^3 = e, (bc)^2 = e \rangle$$

The group is the set of elements

$$\mathbb{S}_3 = \mathbb{D}_3 = \{e, c, c^2, b, bc, bc^2\}$$

2. *Identify these group elements with permutations in the symmetric group* $\mathbb{S}_3$ *where, for example, the identity is denoted*

$$e = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

(Hint: Please note that this identification is not unique. Choose one which is consistent with the group multiplication. It is sufficient to do this for the generators, as long as you can show that the generators obey the relations, the rest of the group elements should follow from simply taking products of the permutations corresponding to the generators. Please, still, list the six permutations corresponding to all six of the group elements.)

3. *Write each $\mathbb{S}_3$ element that you identified for the question above in cycle notation.* For example,

$$e = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = (1)(2)(3)$$

3 Conjugacy classes of $\mathbb{S}_N$

The conjugation of a group element $g \in G$ by another group element $h \in G$ is defined as the product hgh^{-1}. Consider two elements, P and Q, of symmetric group, $\mathbb{S}_N$,

$$P = \begin{bmatrix} 1 & 2 & \ldots & N \\ P(1) & P(2) & \ldots & P(N) \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 2 & \ldots & N \\ Q(1) & Q(2) & \ldots & Q(N) \end{bmatrix}$$

An important fact about symmetric group elements is that the conjugate group element

$$QPQ^{-1}$$

has the same cycle structure as the group element P. The term "same cycle structure" means that its decomposition into cycles results in the same number of cycles of each length. To see this, we can write the permutation as its cycle decomposition:

$$P = (a_1 \ldots a_{\nu_1})(b_1 \ldots b_{\nu_2})(c_1 \ldots c_{\nu_3}) \ldots$$

Alternatively,

$$P = \begin{bmatrix} a_1 & a_2 & \ldots & a_{\nu_1} & b_1 & b_2 & \ldots & b_{\nu_2} & \ldots \\ a_2 & \ldots & a_{\nu_1} & a_1 & b_2 & \ldots & b_{\nu_2} & b_1 & \ldots \end{bmatrix}$$

where we see that in each cycle, the permutation simply performs a cyclic re-ordering of the integers which are in the cycle.

Then, we write Q as

$$Q = \begin{bmatrix} a_1 & \ldots & a_{\nu_1} & b_1 & \ldots & b_{\nu_2} & c_1 & \ldots & c_{\nu_3} & \ldots \\ Q(a_1) & \ldots & Q(a_{\nu_1}) & Q(b_1) & \ldots & Q(b_{\nu_2}) & Q(c_1) & \ldots & Q(c_{\nu_3}) & \ldots \end{bmatrix}$$

and Q^{-1} as

$$Q^{-1} = \begin{bmatrix} Q(a_1) & \ldots & Q(a_{\nu_1}) & Q(b_1) & \ldots & Q(b_{\nu_2}) & Q(c_1) & \ldots & Q(c_{\nu_3}) & \ldots \\ a_1 & \ldots & a_{\nu_1} & b_1 & \ldots & b_{\nu_2} & c_1 & \ldots & c_{\nu_3} & \ldots \end{bmatrix}$$

Finally, we can write the conjugation of P by Q as

QPQ^{-1}
$$= \begin{bmatrix} a_2 & \ldots & a_{\nu_1} & a_1 & \ldots \\ Q(a_2) & \ldots & Q(a_{\nu_1}) & Q(a_1) & \ldots \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \ldots & a_{\nu_1} & \ldots \\ a_2 & \ldots & a_{\nu_1} & a_1 & \ldots \end{bmatrix} \begin{bmatrix} Q(a_1) & \ldots & Q(a_{\nu_1}) & \ldots \\ a_1 & \ldots & a_{\nu_1} & \ldots \end{bmatrix}$$

We see that $Q(a_1)$ is replaced by a_1, then a_1 by a_2, then by a_2 then by $Q(a_2)$, the net result being $Q(a_1) \to Q(a_2)$. Similarly $Q(a_2) \to Q(a_3) \ldots Q(a_{\nu_1}) \to Q(a_1)$. This is a cycle $(Q(a_1) \ldots Q(a_{\nu_1}))$. By repeating this reasoning for the next cycles, we see that

$$QPQ^{-1} = (Q(a_1) \ldots Q(a_{\nu_1})) \, (Q(b_1) \ldots Q(b_{\nu_2})) \, (Q(c_1) \ldots Q(c_{\nu_3})) \ldots$$

We see that the result of the conjugation of a permutation is a permutation with cycles which are of precisely the same length and multiplicity as the cycles in the original permutation. The only difference is that the assignment of the integers to the cycles is changed by the permutation Q.

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Let us make the hypothesis that the set of all possible permutations is equal to the set of all possible inequivalent cycle decompositions of a permutation. The different possible cycle structures of the permutations must be given by all of the partitions of the integer N. A partition of N is an unordered set of positive integers which adds up to N. For example, if $N = 3$, the partitions are

$$\text{partitions of } 3: \quad 1+1+1, \ 1+2, \ 3$$

Thus, we say that the possible cycle decompositions of an element of $\mathbb{S}_3$ can have three 1-cycles, one 1-cycle and one 2-cycle and, finally, one 3-cycle. Then the group elements are given by the inequivalent assignments of the integers 1,2,3 to the cycles with the proviso that the ordering of the cycles is irrelevant and the ordering within a cycle has cyclic symmetry. By this counting, the partition 1+1+1 coincides with three 1-cycles and one group element

$$(1)(2)(3) = e$$

which is the identity. The partition 1+2 coincides with one 1-cycle and one 2-cycle and there are three different group elements

$$(1)(23), \ (2)(31), \ (3)(12)$$

Finally the partition 3 coincides with one 3-cycle and two group elements

$$(123), \ (321)$$

(Convince yourself that any other ordering of 1,2,3 in the 3-cycle is equivalent to one of those listed.)

4 Young diagrams

There is a nice diagrammatic way to enumerate the elements and study representations of $\mathbb{S}_N$. It also plays an important role in representation theory of Lie algebras which we shall study later. To implement it, we consider the partitions of N. These coincide with sets of positive integers, $\lambda_1, \lambda_2, \ldots, \lambda_r$ which satisfy

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$$

and

$$\lambda_1 + \lambda_2 + \ldots + \lambda_r = N$$

Let us consider a particular partition, $\lambda_1, \lambda_2, \ldots, \lambda_r$, of N. This partition coincides with a Young diagram which consists of r rows of boxes, the first row having λ_1 boxes, the second row having λ_2 boxes until the r'th row which must have λ_r boxes arranged in such a way that the resulting array of boxes is top and left-justified.

An example is the partition of the integer $N = 14$ by the integers $8 + 3 + 2 + 1$ which coincides with the Young diagram
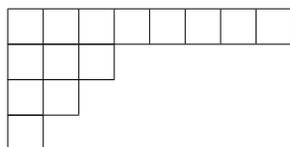
By these rules, each row of boxes is of length less than or equal to the row above it. Each column of boxes is of length less than or equal to the column that is to the left of it. There are always N boxes in total. We call such an array of top-justified and left-justified boxes a "legal Young diagram". It is clear that there is a one-to-one correspondence between partitions of N and the legal Young diagrams with N boxes.

For example, the partitions of 3 what we used to study $\mathbb{S}_3$ were $1 + 1 + 1$, $1 + 2$ and 3. These coincide with the Young diagrams

$$1 + 1 + 1 \ = \ \square\!\!\square\!\!\square \qquad 1 + 2 \ = \ \square \qquad 3 \ = \ \square\square\square \tag{1}$$

7. *Find the Young diagrams corresponding to the partitions of 4.*

5   Young Tableaux

When we assign values to the boxes of a Young diagram, we obtain a Young tableau. Various values are used for various purposes. Here, we will use the integers 1,2,...,N for a Young diagram with N boxes.

We have already developed a way in which this could be useful to us. We could use the set of legal Young diagrams with N boxes to determine the possible cycle decompositions of the elements of $\mathbb{S}_N$. We recall that a possible cycle decomposition is specified by a partition of N, as is a Young diagram. Beyond that, the cycles in the decomposition are assigned integers from the set $1, ..., N$ in such a way that each integer occurs once. All possible assignments, discarding the multiplicity where assignments differ by cyclic permutations of the cycles, result is all group elements with that particular cycle decomposition. Those group elements are a subset of $\mathbb{S}_N$ and it is a complete conjugacy class of $\mathbb{S}_N$.

Instead of the cycles, we could assign the integers to the boxes of a particular Young diagram. This must also enumerate the group elements with a particular cycle decomposition. We simply assign the integers $1, ..., N$, one each to the N boxes in the Young tableau. We drop multiplicities which differ by cyclic permutations of the rows in the tableau. For example, the Young tableaux

1	2	3
4	5	

and

1	2	3
5	4	

and

$$\begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\\hline 5 & 4 \\\cline{1-2}\end{array}$$

correspond to identical cycle decompositions and should be counted as one group element

$$(123)(45)$$

of $\mathbb{S}_5$. This gives a way of determining the conjugacy classes of $\mathbb{S}_N$.

For example, for $\mathbb{S}_3$, we simply study all possible assignments of $1, 2, 3$ to the boxes of the Young diagrams, eliminate those which differ by a cyclic permutation of the rows in the Young diagrams and we would have a list of the group elements in cycle notation,

$$\begin{array}{c}\square \\ \square \\ \square \end{array} \rightarrow \begin{array}{|c|} \hline 1 \\\hline 2 \\\hline 3 \\\hline \end{array} \tag{2}$$

$$\begin{array}{c}\square\,\square \\ \square \end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 2 \\\hline 3 \\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\\hline 1 \\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\\hline 2 \\\cline{1-1}\end{array} \tag{3}$$

$$\square\,\square\,\square \rightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\\hline\end{array} \begin{array}{|c|c|c|}\hline 2 & 3 & 1 \\\hline\end{array} \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\\hline\end{array} \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\\hline\end{array} \begin{array}{|c|c|c|}\hline 2 & 1 & 3 \\\hline\end{array} \begin{array}{|c|c|c|}\hline 1 & 3 & 2 \\\hline\end{array} \tag{4}$$

$$\rightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\\hline\end{array} \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\\hline\end{array} \tag{5}$$

which coincide with the conjugacy classes of $\mathbb{S}_3$.

Another interesting application of Young tableaux is in determining the dimensions of the irreducible representations of $\mathbb{S}_N$. Let us begin with an example, Consider the inequivalent irreducible representations of $\mathbb{S}_3$ which are identical to the inequivalent irreducible representations of $\mathbb{D}_3$. Those are two one-dimensional representations, where the generators are represented by

$$D^{(0)}(b) = 1 \ , \quad D^{(0)}(c) = 1$$
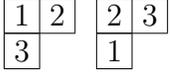$$D^{(1)}(b) = -1 \ , \quad D^{(1)}(c) = 1$$

and one two-dimensional representation which can be written explicitly as the matrices

$$D^{(2)}(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \quad D^{(2)}(c) = \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{bmatrix}$$

It is a fact that each distinct shape of a legal Young diagram with N boxes corresponds to an irreducible representation of $\mathbb{S}_N$. The dimension of that

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representation is equal to the number of standard Young tableaux of that distinct shape.

A standard Young tableau is a legal Young diagram with N boxes and with the integers $1, 2, ..., N$ entered into the boxes so that each integer occurs once, that is in one and only one box of the Young diagram. Moreover, the integers must occur in such a way that those located in any column are strictly increasing as the column is read from top to bottom and the integers in any row are strictly increasing as the row is read from left to right. We can list the standard Young tableaux for $N = 3$ as

Young diagram	Standard Young Tableaux	Representation of $\mathbb{S}_3$
	$\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}$	$D^{(1)}$
	$\boxed{\begin{array}{cc}1&2\\3&\end{array}}\quad\boxed{\begin{array}{cc}2&3\\1&\end{array}}$	$D^{(2)}$
	$\boxed{\begin{array}{ccc}1&2&3\end{array}}$	$D^{(0)}$

$$(6)$$

where, in the third column, we have also indicated the representation for which their number matches the dimension. We see that, in each case, the Young diagram corresponds to a particular representation and the number of standard Young tableaux that are created from that diagram equals the dimension of the representation.

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8. *Use the standard Young tableaux to find the dimensions of the inequivalent irreducible representations of $\mathbb{S}_4$.*

An alternative to the counting of standard Young tableaux of a certain shape, a way to determine the dimension of the representation of $\mathbb{S}_N$ corresponding to a particular shape of N-boxed Young diagram is called the "hook length formula".

There is a hook corresponding to every box in a Young diagram. For a given box, it is defined as the set of boxes in the same row but to the right of it, plus the set of boxes in the same column but below it plus the box itself. The hook is assigned a number which is the total number of those boxes. A

simple example is

3	1
1	

where, in each box, we have written the hook length, the total number of boxes in the hook corresponding to that box. Another example is

7	5	4	3	1
5	3	2	1	
1				

Then, the dimension of the corresponding representation of $\mathbb{S}_N$ is

$$\dim \left[\text{rep of } \mathbb{S}_N \right] \; = \; \frac{N!}{\prod_{\text{hooks}} \text{hook lengths}} \tag{7}$$

9. *Confirm that the hook formula (7) reproduces the dimensions of the irreducible representations of $\mathbb{S}_4$ which you found by using the multiplicity of standard Young tableaux in problem 8 above.*

6 $\mathbb{S}_N$ and the Braid Group $\mathbb{B}_N$

A commonly used presentation of the symmetric group $\mathbb{S}_N$ is

$$\mathbb{S}_N = \langle \sigma_1, \sigma_2, \ldots, \sigma_{N-1} | \sigma_i^2 = e, \sigma_i \sigma_j = \sigma_j \sigma_i \forall |i-j| > 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$$

This presentation uses the generators $\sigma_1, \sigma_2, \ldots, \sigma_{N-1}$ where σ_i is an interchange of the object in position i with the object in position $i + 1$. These generators clearly obey the relations $\sigma_i^2 = e$ and interchanges which do not overlap must commute with each other $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$. Neighbouring interchanges also obey the braid relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, ..., N - 1$. The validity of the latter is illustrated in figure 1.

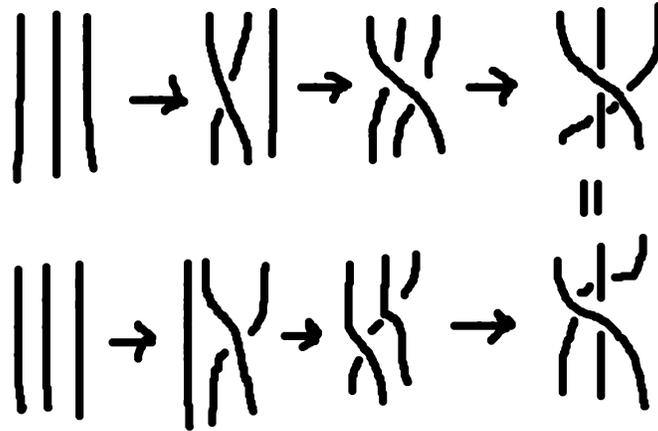Figure 1: The figure depicts the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for three consecutive generators of the braid group. It also applies to the generators of the permutation group if we do not distinguish whether the lines pass over or under each other.

As a concrete example. in terms of the generators σ_1 and σ_2, the elements of $\mathbb{S}_3$ are
$$\mathbb{S}_3 = \{e, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$$
The relations can be used to collapse all other products of the generators to one of these.

10. *Identify the group elements $S^3 = \{e, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$ that are found using the presentation*

$$\mathbb{S}_3 = \langle \sigma_1, \sigma_2 | \sigma_1^2 = e, \sigma_2^2 = e, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

with the elements $\mathbb{S}_3 = \{e, c, c^2, b, bc, bc^2\}$ which are found by using the presentation

$$\mathbb{S}_3 = \langle b, c | b^2 = e, c^3 = e, (bc)^2 = e \rangle$$

An important infinite discrete group called the braid group $\mathbb{B}_N$ uses the same generators as the symmetric group, $\sigma_1, \sigma_2, \ldots, \sigma_{N-1}$ where σ_i is an oriented interchange of the object in position i with the object in position

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$i + 1$, or a braiding of the two where, when they are exchanged, the i'th object passes "above" the $i + 1$'th object. They also obey the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, ..., N - 1$ which was illustrated in figure 1, where it was anticipated that the sense in which permutations of neighbours are oriented matters. In the figure, σ_i interchanged object i and object $i+1$ by passing i over $i + 1$, as drawn in the figure.

The difference from the symmetric group is that these generators do not obey the identity $\sigma_i^2 = e$. Therefore, they each generate an infinite set of group elements, $\{\sigma_i, \sigma_i^2, \sigma_i^3, ...\}$. Moreover, to form a group, they must have inverses which now must be included separately as negative powers, so that for each generator, there is the set of group elements $\{\sigma_i, \sigma_i^{-1}, \sigma_i^2, \sigma_i^{-2}, \sigma_i^3, \sigma_i^{-3}, ...\}$. In fact, if we include the identity, the group $\mathbb{B}_2$ has elements

$$\mathbb{B}_2 = \{e = \sigma^0, \sigma, \sigma^{-1}, \sigma^2, \sigma^{-2}, \sigma^3, \sigma^{-3}, ...\}$$

This is an Abelian group which is isomorphic to the additive group of the integers.

We note that, in quantum mechanics, systems of many identical and indistinguishable particles have completely symmetric or antisymmetric wavefunctions in the case where they are bosons or fermions, respectively. These correspond to the one dimensional representations of $\mathbb{S}_N$ which are represented by the Young tableaux

$$\boxed{1}\boxed{2}\boxed{3}\ ..N.$$

and

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline . \\ \hline . \\ \hline . \\ \hline N \\ \hline \end{array}$$

respectively. Since these are unique standard Young Tableaux, they both correspond to one-dimensional representations.

11. *Show that the completely symmetric and the completely anti-symmetric representations that we discussed above are the only one dimensional representations of $\mathbb{S}_N$.*

Other types of particles, carrying representations of $\mathbb{S}_N$ other than the two one-dimensional representations whose Young diagrams are displayed above, are said to have "parastatistics" . Since you have shown that their representations necessarily have dimensions greater than one, their quantum states, including the many-body ground state, must be degenerate.

Because of topology, in two space dimensions, particles can have more sophisticated statistics where a many-particle wave-function can transform under a representation of the braid group, $\mathbb{B}_N$, rather than the symmetric group $\mathbb{S}_N$.

12. *Find some of the possible one-dimensional representations of $\mathbb{B}_N$.*