PHYS 521D TUTORIAL 3

The Symplectic Lie Algebra sp(2N)

Consider a classical mechanical system where the N generalized coordinates, $q_1, q_2, ..., q_N$ and N generalized momenta $p_1, ..., p_N$ which obey the Poisson bracket

$$\{q_i, p_j\} = \delta_{ij} \ , \ i, j = 1, 2, ..., N$$

The space of all possible values of q_i and p_i is a $2N$-dimensional space called phase space. It is sometimes useful to think of it has having $2N$ coordinates

$$x_a \equiv (q_1, ..., q_N, p_1, ..., p_N)$$

In this notation, the Poisson bracket is given by the expression

$$\{x_b, x_b\} = J_{ab}$$

where J is a matrix

$$J = \begin{bmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{bmatrix}$$

where 0 and $\mathcal{I}$ stand for the $N \times N$ zero and unit matrices, respectively. We might ask the question as to what sort of linear transformations of the coordinates

(1) $$x_a \ \rightarrow \ \tilde{x}_a = S_a{}^b x_b$$

result in new coordinates which have the same Poisson bracket,

$$\{\tilde{x}_b, \tilde{x}_b\} = J_{ab}$$

Here, $S_a{}^b$ is a $2N \times 2N$ matrix whose entries are real numbers.

(1) *Show that the matrices in the linear transformation (1) must obey the equation*

$$SJS^t = J$$

Here S^t is the transpose of the matrix S.

(2) *Show that the set of all $2N \times 2N$ real matrices S which obey the equation $SJS^t = J$ form a group.*

This group is called the symplectic group, SP(2N,R), the R appearing since the matrices are real.

Now that we have a defining representation of our Lie group, let us examine the Lie algebra. For this purpose, we consider group elements in a small neighbourhood of the identity

$$S = \mathcal{I} + iK$$

The matrices K are elements of the Lie algebra sp(2N,R) corresponding to the Lie group SP(2N,R).

(3) *Show that a matrix $K \in$sp(2N,R) is of the form*

$$K = -i \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \tag{2}$$

where the $N \times N$ blocks are real matrices which have the properties $B = B^t$, $C = C^t$ and A is unrestricted.

(4) *Demonstrate that the set of all matrices K of the form given in equation (2) indeed form a Lie algebra.*

(5) *What is the dimension of sp(2N)?*

Now we come to a difficulty. The elements of the Lie algebra in equation (2) are not Hermitian matrices. In fact this is not a compact Lie algebra and it falls outside of the real of the compact semi-simple Lie algebras that we have developed the formalism of roots, weights, etc. for. There is, however, a convenient trick for understanding it. What we do is we consider a compact Lie algebra which has very similar properties and which is a compact semi-simple Lie algebra and we study it.

A symptom of the non-compactness of the matrices in (2) would be seen immediately when we try to write a generic K as a sum of real numbers times basis matrices. Some of the basis matrices would have to be anti-Hermitian. The alternative algebra that we will study is actually obtained from this one by simply multiplying the anti-Hermitian matrices by a factor of i or $-i$ so that all of the matrices are Hermitian. Once we find representations of the resulting compact Lie algebra, we can do the inverse redefinition of generators to find the representations of the real symplectic algebra if we want to.

To implement our "compactification" of the algebra, it is most efficient to just start again and ask the question, what unitary matrices U have the property

$$UJU^t = J \ , \ UU^\dagger = \mathcal{I} \tag{3}$$

(6) *Show that a unitary matrix satisfying (3) indeed form a group.* We will call this Lie group SP(2N), as distinct from SP(2N,R).

Now, we examine transformations in the vicinity of the identity

$$U = \mathcal{I} + iK$$

Since U is unitary, K must be Hermitian.

(7) *Show that, if U satisfies (3), K has the general form*

$$K = \begin{bmatrix} A & B \\ B^\dagger & -A \end{bmatrix} \ , \ B = B^t \ , \ A = A^\dagger \tag{4}$$

(8) *Show that the set of all complex matrices of the form (4) comprise a Lie algebra.* We will call this Lie algebra sp(2N).

(9) *What is the dimension of the Lie algebra of the K's of the form (4). Does this unitary symplectic algebra have the same dimension as sp(2N,R)?*

The most efficient way to understand the properties of the algebra is to begin with the Cartan subalgebra. We can do this by simply picking the diagonal matrices of the appropriate dimension and with the appropriate properties. The diagonal matrices of the form in equation (4) have $B = 0$ and A diagonal and we simply need to choose an appropriate linearly independent and complete set of such matrices . These could be the N matrices (with no summation over the index b)

$$[H^a]_{bc} = \delta_{bc}\delta^a_b - \delta_{bc}\delta^{a+N}_b \ , \ a = 1, 2, ..., N; \ b, c = 1, 2, ..., 2N$$

These are normalized so that $\mathrm{Tr}(H^a H^b) = 2\delta^{ab}$.

There are N matrices in the Cartan subalgebra. This Lie algebra has rank N, the same rank, as sp(2N,R). Thus, the unitary symplectic algebra has the same dimension and rank as sp(2N,R) and it can be thought of as a complexification of that algebra.

(10) *Show that the weights of this 2N-dimensional defining representation are the 2N vectors*

$$\pm\hat{e}^i$$

where $\hat{e}^i$ is a unit vector aligned the the i'th Cartesian direction in N-dimensional Euclidean space.

The roots are the weights of the adjoint representation. They are also the vectors which are added to weights of any representation in order to find other weights of that same representation. We have identified the weights of the defining representation. The distances between these weights are potential roots. Remember that the total number of roots must equal the dimension D of the algebra. Also remember that the number of zero roots is equal to the rank of the algebra, in this case, $r = N$. Thus there must be $D - r$ nonzero roots.

(11) *Find the roots.*
(12) *Find the positive roots.*
(13) *Find the simple roots.*
(14) *Find the Dynkin diagram.*
(15) *Find the fundamental weights.*
(16) Are the defining or the adjoint representation amongst the list of fundamental representations?
(17) *Find all of the weights of the two fundamental representations of sp(4).*