Tutorial 4: The Renormalization Group and the Ising Model

The classical Ising model on a hyper-cubic lattice has a Hamiltonian given by the expression

$$H = -J \sum_x \sum_{j=1}^{D} [\sigma(x)\sigma(x + \hat{e}_j) + \sigma(x)\sigma(x)] - \sum_x B(x)\sigma(x) \tag{1}$$

where

$$x = \sum_{a=1}^{D} \; n_a \hat{e}^a \tag{2}$$

with $\hat{e}^a$ the unit vector aligned with the a'th Cartesian direction in D-dimensional space and $(n_1, ..., n_D)$ integers. n^a vary over the integers, the positions x sweep out the sites of a D-dimensional hyper-cubic lattice. Here, we are working in a system of distance units where the lattice spacing equals one. If this were not so, there would be an additional factor if $\tilde{c}$, the lattice constant on the right-hand-side of equation (2).

The spin degree of freedom living at each site of the lattice, $\sigma(x)$, take on the values +1 and −1. The constants J and B in the Hamiltonian have the dimensions of energies. Usually, $B(x)$ is taken to be an x-independent constant. We will indeed do this eventually, in fact after a while we will set B to zero. Here, for a brief while, we keep it x-dependent since derivatives of the logarithm of the partition function by $B(x)$ with various values of x can then be used to generate correlation functions. We will see how this works shortly.

We will assume that J is a positive real number. In this case, the model is Ferromagnetic in that the lowest energy states of the first term in the Hamiltonian (1) have all of the spins aligned. If the external field $B(x)$ is switched off, the model has Z_2 symmetry in that the Hamiltonian is left invariant by the transformation $\sigma(x) \to -\sigma(x)$ for all of the spins.

The energy function (1) contains a term,

$$-J \sum_x \sum_{j=1}^{D} [\sigma(x)\sigma(x)] = -VJD \tag{3}$$

which has been added to it for technical reasons – to make the first part of the Hamiltonian negative definite. We will need this in order to do the Gaussian transform. Adding an overall constant to the energy does not change anything as far as the statistical mechanics computations that we will do are concerned. In particular, it multiplies both the Boltzmann weight and the partition function by the factor $e^{\frac{VJD}{k_BT}}$ which cancels in the computations of all expectation values except the expectation value of the energy, where it shifts the energy by the innocuous constant in equation (3).

The partition function is found by averaging the Boltzmann weight, e^{-H/k_BT} over all of the possible spin configurations,

$$Z[T,N,B] = \sum_{\text{spins}} e^{\frac{J}{k_BT} \sum_x \sum_{j=1}^{D} [\sigma(x)\sigma(x+\hat{e}_j) + \sigma(x)\sigma(x)] + \sum_x \frac{B(x)}{k_BT}\sigma(x)}. \tag{4}$$

We know about the exact solutions of the Ising model in $D = 1$ and $D = 2$. In particular, in $D = 2$ it has a second order phase transition. We expect that this behaviour persists in higher dimensions and our goal is to study it. For that reason, we are interested in $D > 2$ in the following.

We have used a Gaussian transformation to rewrite its partition function as that of a lattice field theory with a lattice field $\phi(x)$. Unlike the spins which have values ± 1, the lattice fields take on values in the real numbers. This will be needed in order to set up our use of the renormalization group which we will get to shortly.

A collection of formulae for the Ising model in terms of the lattice fields

$\phi(x)$ is as follows:

$$Z[T,N,B] = \frac{\int_{-\infty}^{\infty} \prod_x d\phi(x) \exp\left(-S[\phi]\right)}{\left[\sqrt{\det 2\pi\Delta}\right]} \tag{5}$$

$$S[\phi] = S_0[\phi] + V[\phi] \tag{6}$$

$$S_0[\phi] = \frac{1}{2}\sum_{x,y} \phi(x)\Delta^{-1}(x,y)\phi(y) \tag{7}$$

$$\Delta(x,y) = \frac{J}{k_B T}\sum_{a=1}^{D}\left[\delta_{x,y+\hat{e}_a} + \delta_{x,y-\hat{e}_a} + 2\delta_{x,y}\right] \tag{8}$$

$$= \int_{\Omega_B} \frac{d^D p}{(2\pi)^D} e^{ip\cdot(x-y)} \frac{J}{k_B T}\sum_{a=1}^{D}\left[2\cos p_a + 2\right] \tag{9}$$

$$\Delta^{-1}(x,y) = \int_{\Omega_B} \frac{d^D p}{(2\pi)^D} e^{i\vec{p}\cdot(x-y)} \frac{1}{\left[\frac{J}{k_B T}\sum_{a=1}^{D} 2[\cos p_a + 1]\right]} \tag{10}$$

$$\Delta^{-1}(x,y) = \int_{\Omega_B} \frac{d^D p}{(2\pi)^D} e^{i\vec{p}\cdot(x-y)} \frac{\frac{K_B T}{4JD}}{\left[1 - \sum_{a=1}^{D} \frac{1}{D}\sin^2\frac{p_a}{2}\right]} \tag{11}$$

$$\tag{12}$$

$$V[\phi] = \sum_x V(\phi(x)) \tag{13}$$

$$V(\phi(x)) = -\ln 2\cosh\left(\phi(x) + \frac{B(x)}{k_B T}\right) \tag{14}$$

The leading terms in the Taylor expansion of the potential are

$$V(\phi) = -\frac{1}{2}\phi^2 + \frac{1}{12}\phi^4 \;+\; \ldots \;-\; \ln 2$$

We have left the B-field non-zero so far, in fact we have made it position dependent. This is so that we can derive formulae for the expectation value of the spin, that is, the magnetization density and for the correlation function

$$m(x) \equiv < \sigma(x) >= \sum_y \Delta^{-1}(x,\vec{y}) < \phi(y) > \tag{15}$$

$$\chi(x,y) \equiv < \sigma(x)\sigma(y) >_C = \sum_{wz} \Delta^{-1}(x,w)\langle\, \phi(w)\phi(z)\, \rangle_C \Delta^{-1}(z,y) - \Delta^{-1}(x,y) \tag{16}$$

The subscript C on the correlation function means "connected correlation function" defined by

$$< \sigma(x)\sigma(y) >_C \equiv < \sigma(x)\sigma(y) > - < \sigma(x) >< \sigma(y) > \tag{17}$$

1 Partition function as a generating function for correlation functions

1. *Use equations (5)-(14) to demonstrate that the expressions for the one- and two-point functions that are given in equations (15) and (16) can be obtained from the definitions*

$$F[T,N,B] = -k_B T \ln Z[T,N,B] \tag{18}$$

$$m(x) = -\frac{\partial}{\partial B(x)} F[T,N,B] \tag{19}$$

$$\chi(x,y) = -(k_B T)\frac{\partial^2}{\partial B(x)\partial B(y)} F[T,N,B] \tag{20}$$

respectively.

Once we have these equations for correlation functions, we shall set $B = 0$. We can confirm that, once $B = 0$, the model has the Z_2 symmetry where the integrations which compute the partition function are invariant under the change of variables $\phi(x) \to -\phi(x)$. When the temperature is greater than the critical temperature T_C this symmetry should average the magnetization density, $< \sigma(x) >$ to zero. On the other hand, in the low temperature regime, the Z_2 symmetry is spontaneously broken and $< \sigma(x) >$ is non-zero.

One way to see that this can happen is to simply study the partition function in equation (5) in the large T limit or the small T limit.

2 High T limit

Let us begin with the large T limit. We want to study the behaviour of the integral (here, we have set $B = 0$)

$$Z[T,N] = \frac{\int_{-\infty}^{\infty} \prod_x d\phi(x) \exp\left(-S[\phi]\right)}{\left[\sqrt{\det 2\pi\Delta}\right]}$$

with

$$S[\phi] = \frac{1}{2}\sum_{x,y} \phi(x)\Delta^{-1}(x,y)\phi(y) + \sum_x V(\phi(x))$$

in the limit where T is large. In particular, it is instructive to ask how the limit of large T of the integral above reproduces the large T limit of the Ising model partition function that is written in equation (4)

$$\lim_{T\to\infty} Z[T,N] = \lim_{T\to\infty} \sum_{\text{spins}} e^{\frac{J}{k_BT}\sum_x \sum_{j=1}^D [\sigma(x)\sigma(x+\hat{e}_j)+\sigma(x)\sigma(x)]} = \sum_{\text{spins}} 1 \ = 2^N$$

We recall that, from equation (12),

$$\Delta^{-1}(x,y) = \int_{\Omega_B} \frac{d^Dp}{(2\pi)^D} e^{i\vec{p}\cdot(x-y)} \frac{\frac{K_BT}{4JD}}{\left[1-\sum_{a=1}^D \frac{1}{D}\sin^2\frac{p_a}{2}\right]}$$

and

$$V(\phi(x)) = -\ln 2\cosh(\phi(x))$$

When T is very large, the first term, quadratic term in $S[\phi]$ is large in that it has a large coefficient due to the factor $\frac{K_BT}{4JD}$.

One could remove the factor of T from the quadratic term by changing variables in the integrals over $\phi(x)$, specifically, $\phi(x)\to\sqrt{\frac{4JD}{K_BT}}\phi(x)$.

The integration measure would get a factor of $\left(\frac{4JD}{K_BT}\right)^{\frac{N}{2}}$ and the quadratic term becomes

$$\frac{1}{2}\sum_{x,y}\phi(x)\frac{4JD}{K_BT}\Delta^{-1}(x,y)\phi(y)$$

which is independent of T. The potential energy term becomes

$$V\left(\sqrt{\frac{4JD}{K_BT}}\phi(x)\right) = -\ln 2\cosh\left(\sqrt{\frac{4JD}{K_BT}}\phi(x)\right) \approx -\ln 2$$

In the large T limit, this potential energy term becomes a ϕ-independent constant and only the Gaussian terms are left.

The partition function is then given by the formula

$$\lim_{T\to\infty} Z[T,N] = 2^N\left(\frac{4JD}{K_BT}\right)^{\frac{N}{2}} \frac{\int_{-\infty}^{\infty}\prod_x d\phi(x)\exp\left(-\frac{1}{2}\sum_{x,y}\phi(x)\frac{4JD}{K_BT}\Delta^{-1}(x,y)\phi(y)\right)}{\left[\sqrt{\det 2\pi\Delta}\right]}$$

2. *Perform the Gaussian integral in the above formula to show that it reproduces the expected result*

$$\lim_{T\to\infty} Z[T,N] = 2^N$$

3. *Use equations (19) and (20) to show that*

$$\lim_{T\to\infty} m(x) = 0$$

and

$$\lim_{T\to\infty} \chi(x,y) = 0$$

Extracurricular question (Not mandatory and not for marks): *The above formula presumably tells us that the correlation length goes to zero as the temperature goes to infinity. Can you estimate the temperature dependence of the correlation length in the large T limit?*

3 Low T limit

Let us now consider the low temperature limit, $T \to 0$. Again, it is instructive to ask how the limit of small T of the integral formula for the partition function reproduces the small T limit of the Ising model partition function that is written in equation (4)

$$\begin{aligned}\lim_{T\to 0} Z[T,N] &= \lim_{T\to 0} \sum_{\text{spins}} e^{\frac{J}{k_BT}\sum_x \sum_{j=1}^D [\sigma(x)\sigma(x+\hat{e}_j)+\sigma(x)\sigma(x)]} \\ &= \lim_{T\to 0} e^{\frac{2DJ}{k_BT}N} \end{aligned} \tag{21}$$

where we have assumed that only one spin configuration, either all up, or all down contributes. In order to choose which would contribute, we imagine turning on a small B term which biases the spin to have the same sign as B. Then the zero temperature spin configuration is unique and we turn off the B term to produce (21).

4. *Show that, in the low temperature limit, the expression for the partition function in equation (5) reduces to the problem of computing the*

integral

$$\lim_{T\to 0} Z[T,N] =$$

$$= \lim_{T\to 0}\left(\frac{4JD}{K_BT}\right)^{\frac{N}{2}} \frac{\int_{-\infty}^{\infty}\prod_x d\phi(x) e^{-\frac{1}{2}\sum_{x,y}\phi(x)\frac{4JD}{K_BT}\Delta^{-1}(x,y)\phi(y)+\sum_x \ln 2\cosh\left(\sqrt{\frac{4JD}{K_BT}}\phi(x)\right)}}{\left[\sqrt{\det 2\pi\Delta}\right]}$$

$$= \lim_{T\to 0}\left(\frac{4JD}{K_BT}\right)^{\frac{N}{2}} \frac{\int_{-\infty}^{\infty}\prod_x d\phi(x) e^{-\frac{1}{2}\sum_{x,y}\phi(x)\frac{4JD}{K_BT}\Delta^{-1}(x,y)\phi(y)+\sum_x \sqrt{\frac{4JD}{K_BT}}|\phi(x)|}}{\left[\sqrt{\det 2\pi\Delta}\right]}$$

The last integral in the line above is difficult because of the absolute value sign in the last term in the exponent. If that absolute value sign were absent, it would be a relatively easy off-set Gaussian integral. We can see that, when T is very small, the integrand is concentrated in two regions, $\phi \sim 1/\sqrt{T}$ and $\phi \sim -1/\sqrt{T}$. Even a small bias of the integrand, say with an infinitesimal B-field term, would favour one of these regions over the other so that the entire integration is concentrated there. In that circumstance, we would obtain the same result as if we remove the absolute value sign, that is,

$$\lim_{T\to 0} Z[T,N] =$$

$$= \lim_{T\to 0}\left(\frac{4JD}{K_BT}\right)^{\frac{N}{2}} \frac{\int_{-\infty}^{\infty}\prod_x d\phi(x) e^{-\frac{1}{2}\sum_{x,y}\phi(x)\frac{4JD}{K_BT}\Delta^{-1}(x,y)\phi(y)+\sum_x \sqrt{\frac{4JD}{K_BT}}\phi(x)}}{\left[\sqrt{\det 2\pi\Delta}\right]}$$

5. *Show that the above formula gives the result*

$$\lim_{T\to 0} Z[T,N] = \lim_{T\to 0}\frac{\left[\sqrt{\det 2\pi\Delta}\right] e^{\frac{1}{2}\sum_{x,y}\Delta(x,y)}}{\left[\sqrt{\det 2\pi\Delta}\right]}$$

and that this result matches what we expected from equation (21).

6. *Use the formulae for the low T limit that we found in the paragraphs above and our expression for the magnetization density from equation (15),*

$$\lim_{T\to 0} m(x) = \lim_{T\to 0}\sum_y \Delta^{-1}(x,y)\frac{\int_{-\infty}^{\infty}\prod_x d\phi(x)\exp\left(-S[\phi]\right)\phi(y)}{\int_{-\infty}^{\infty}\prod_x d\phi(x)\exp\left(-S[\phi]\right)}$$

to show that the low temperature limit is magnetized,

$$\lim_{T\to 0} m(x) = 1$$

This means that, in the low T limit, the Z_2 symmetry is spontaneously broken.

4 Recovering mean field theory

Before we apply the renormalization group to this model it is instructive to try to understand why we need it. To see this, we could attack the problem of computing the partition function by attempting to do the integral by saddle point technique. This is not a controlled approximation in that, away from the high or low temperature limits, there is no small parameter which we are expanding in, and thus there is no guarentee that the expansion is accurate. All we can do is hope that it is self-consistent in that the result turns out to be a convergent series.

To implement the saddle point technique, we take the expression for the partition function from equation (5)

$$Z[T,N] = \frac{\int_{-\infty}^{\infty} \prod_x d\phi(x) \exp\left(-S[\phi]\right)}{\left[\sqrt{\det 2\pi\Delta}\right]}$$

where

$$S[\phi] = \frac{1}{2}\sum_{x,y} \phi(x)\Delta^{-1}(x,y)\phi(y) - \sum_x \ln 2\cosh\phi(x)$$

and we change the integration variable by the transformation $\phi(x) \to \bar{\phi} + \phi(x)$. Then we expand $S[\bar{\phi}+\phi]$ to quadratic order in ϕ, drop the term that is linear in ϕ and neglect all terms that are of higher orders in ϕ. This gives an approximation to the integral as

$$Z[T,N] \approx$$

$$\approx \frac{\int_{-\infty}^{\infty} \prod_x d\phi(x) \exp\left(-S[\bar{\phi}] - \frac{1}{2}\sum_{x,y}\phi(x)\left[\Delta^{-1}(x,y) - (1-\tanh^2\bar{\phi})\delta_{x,y}\right]\phi(y) + \ldots\right)}{\left[\sqrt{\det 2\pi\Delta}\right]}$$

where $\bar{\phi}$ is to be determined so that $S[\bar{\phi}]$ is at a minimum. The ellipses stand for higher order corrections which could be included if we kept higher than quadratic order terms in the expansion of the action in ϕ.

7. *Show that the formulae above give an approximation to the free energy*

$$F[T,N] = \\ = k_B T\left[S[\bar\phi] + \frac{1}{2}\mathrm{Tr}\ln\left[\Delta^{-1}(x,y) - (1-\tanh^2\bar\phi)\delta_{x,y}\right] - \frac{1}{2}\mathrm{Tr}\ln\left[\Delta^{-1}(x,y)\right] + \ldots\right] \tag{22}$$

where $\bar\phi$ is determined by finding the infimum of $S[\phi]$. We expect that $\bar\phi$ is independent of x. This is equivalent to assuming lattice translation invariance.

8. *Show that, with the assumption that $\bar\phi$ is constant, the approximate expression for the free energy is*

$$F[T,N] = \\ = k_B T N\left[\frac{1}{2}\frac{k_B T}{4DJ}\bar\phi^2 - \ln 2\cosh\bar\phi + \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\ln\left[1 - \Delta(p)(1-\tanh^2\bar\phi)\right]\right] \tag{23}$$

where $\Delta(p)$ is the lattice Fourier transform of $\Delta(x,y)$ and where $\bar\phi$ is determined by the formula

$$0 = \frac{\partial}{\partial\bar\phi}\left[\frac{1}{2}\frac{k_B T}{4DJ}\bar\phi^2 - \ln 2\cosh\bar\phi\right]$$

One might recall, at this point, that these are essentially the same formulae that we encountered for the infinite ranged Ising model which was solved by **mean field theory**.

This expression is intended to be an approximation which, if valid, has the first terms dominant, the second term with the logarithm a small correction and the remaining terms, represented by ellipses an even smaller correction.

9. *Show that, when T is sufficiently close to T_C and $\bar\phi$ is small,*

$$\bar\phi = \begin{cases} 0 & T > T_C \\ \pm\sqrt{3\left(1-\frac{T}{T_C}\right)} & T < T_C \end{cases}$$

and, when $T \sim T_C$,

$$F[T,N] = \begin{cases} k_B T N \left[0 + \text{constant} \cdot (T - T_C)^{\frac{D}{2}} + \ldots \right] & T > T_C \\ k_B T N \left[-\frac{3}{4} \left(1 - \frac{T}{T_C} \right)^2 + \text{constant}' \cdot (T_C - T)^{\frac{D}{2}} + \ldots \right] & T < T_C \end{cases}$$

where the ellipses denote terms that are either constants, or go to zero faster than the terms that are already included as $T = T_C$. The first term in the free energy is the prediction of mean field theory. The second term is the correction from the Gaussian integral. We see that, when $D \geq 4$, the correction from the Gaussian integral is indeed irrelevant to the critical behaviour which is then completely determined by the discontinuity of the second derivative by temperature of the first term. We could interpret this as telling us that our computation is accurate and that the critical behaviour is described by mean field theory.

On the other hand, when $D < 4$ the Gaussian correction is dominant near T_C. We take this to mean that the approximation that we are making is failing. The Gaussian correction has a bigger magnitude than the leading order behaviour and it dominates the critical behaviour. This is a symptom of poor convergence of our approximate computation. It is in this regime where a more sophistocated approach is needed.

5 The renormalization group

The strategy of the renormalization group is tied to the notion that it is the very long wavelength (small wave-number) degrees of freedom which are responsible for the critical behaviour. Understanding their behaviour will be the key to finding critical exponents, for example. The renormalization group gives us a way of focusing on those large wavelength degrees of freedom. It begins by eliminating the largely irrelevant very short wave-length (large wave-number) degrees of freedom and leaves behind an effective field theory for the long-wavelength ones.

5.1 Renormalization group step I: Eliminating short wave-length modes

To implement the renormalization group, we first introduce a cutoff Λ and we decompose the lattice field into small and large wave-number degrees of freedom by using its Fourier transform,

$$\phi(x) = \phi^{<}(x) + \phi^{>}(x) \tag{24}$$

$$\phi(x) = \int \frac{d^D k}{(2\pi)^{D/2}} e^{i\ kx} \phi(\vec{k}) \tag{25}$$

$$\phi^{<}(x) = \int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^{D/2}} e^{i\ kx} \phi(\vec{k}) \tag{26}$$

$$\phi^{>}(x) = \int_{|k|>\Lambda, k\in\Omega_B} \frac{d^D k}{(2\pi)^{D/2}} e^{ikx} \phi(k) \tag{27}$$

$$\Lambda << \pi \tag{28}$$

where

$$\Omega_B = \{k = (k_1, ..., k_D) \mid -\pi < k_a \leq \pi\} \tag{29}$$

is the Brillouin zone for a hyper-cubic lattice in a system of distance units where the lattice constant is equal to one.

The splitting of the field $\phi(x)$ into large and small wave-number fields in equations (26) and (27) is convenient since the integration measure in the set of integrals that must be done to find the partition function also splits up in a convenient way,

$$\prod_x d\phi(x) = \prod_x d\phi^{<}(x) \prod_x d\phi^{>}(x) \tag{30}$$

$$\prod_x d\phi^{<}(x) \equiv \prod_{|k|<\Lambda} d\phi(k) \tag{31}$$

$$\prod_x d\phi^{>}(x) \equiv \prod_{|k|>\Lambda, k\in\Omega_B} d\phi(k) \tag{32}$$

Notice that an expression like $\int \prod_{|k|<\Lambda} d\phi(k)$ which occurs in equation (30) is a product over a continuously infinite set of integrals. It is the measure for a functional integral and integration with a measure like $\int \prod_{|k|<\Lambda} d\phi(k)$ is a functional integral. We will not need to know very much about functional integration. One thing that we will do is a Gaussian integral. Such an integral is defined as

$$\int [d\phi^>] e^{-\frac{1}{2}\sum_{x,y}\phi^>(x)\Delta^{-1}(x,y)\phi^>(y)} = e^{-\frac{1}{2}V\int_{k>\Lambda, k\in\Omega_B}\frac{d^Dp}{(2\pi)^D}\ln[\Delta^{-1}(p)/2\pi]} \tag{33}$$

$$\int [d\phi^<] e^{-\frac{1}{2}\sum_{x,y}\phi^<(x)\Delta^{-1}(x,y)\phi^<(y)} = e^{-\frac{1}{2}V\int_{k<\Lambda}\frac{d^Dp}{(2\pi)^D}\ln[\Delta^{-1}(p)/2\pi]} \tag{34}$$

We remember that the Fourier transform can be viewed as a unitary change of basis in a vector space. The integration over the components of vectors in the two bases, $\prod_x \int d\phi(x)$ and $\prod_{\vec{k}} \int d\phi(\vec{k})$, then differ by a Jacobian factor which is one for the Fourier transformation as we have normalized it.

$$\int [d\phi] = \prod_x \int_{-\infty}^{\infty} d\phi(x) = \prod_k \int_{-\infty}^{\infty} d\phi(k)$$

$$= \prod_{k\le\Lambda} \int_{-\infty}^{\infty} d\phi(k) \;\cdot\; \prod_{k>\Lambda, k\in\Omega_B} \int_{-\infty}^{\infty} d\phi(k) \equiv \int [d\phi^<] \int [d\phi^>]$$

Then we imagine that we integrate out the large wave-number degrees of freedom, $\phi^>(x)$, to obtain an "effective action" for the small wave-number degrees of freedom, $\phi^<(x)$, which is defined by

$$\mathbf{S}_{\text{eff}}[\phi^<] = -\ln\left\{\frac{\int [d\phi^>] \exp\left(-S[\phi^< + \phi^>]\right)}{\left[(2\pi)^{N/2}\sqrt{\det\Delta}\right]}\right\} \tag{35}$$

If we have done the integral above, the problem that would remain to be solved is the remaining integration over the small wave-number degrees of freedom, that is, to compute the partition function from

$$Z[T, N, B] = \int [d\phi] \exp\left(-\mathbf{S}_{\text{eff}}[\phi]\right) \tag{36}$$

where we have dropped the superscript from ϕ but we must remember that the cutoff is now Λ and the field contains modes with wave-vectors only in the region $0 \le |\vec{k}| \le \Lambda$ which is a very small subset of the Brillouin zone. Implicit in this is a restriction as to what we can use the effective field theory for. In the effective field theory, we can only ever compute correlation functions of fields whose wave-vectors have magnitudes smaller than the cutoff Λ. Computing these small wave-number (or large wavelength) correlation functions is sufficient for our purpose, which is to study the critical behaviour.

5.2 Renormalization group step 2: Resetting the scale

Once we have done step 1, we have the problem of computing the partition sum

$$Z[T,N,B] = \int [d\phi] \exp\left(-\mathbf{S}_{\text{eff}}[\phi]\right) \tag{37}$$

where the variables $\phi(x)$ now have the cutoff Λ, that is, the field is defined by

$$\phi(x) = \int_{k<\Lambda} \frac{d^D p}{(2\pi)^D} e^{ipx} \phi(p)$$

The next task is to reset the resolution of the system so that the cutoff, which is $\Lambda << 1$, appears in our new resolution as if the cutoff is 1. We do this by the replacement

$$\phi(x) \quad = \quad \Lambda^{\frac{D-2}{2}} \zeta \tilde{\phi}(\Lambda x)$$

To see what this replacement does, let us consider a monomial that might occur in $\mathbf{S}_{\text{eff}}[\phi]$,

$$\sum_x \phi^n(x)$$

which would then be replaced by

$$\Lambda^{n\frac{D-2}{2}-D} \zeta^n \tilde{\sum_x} \Lambda^D \tilde{\phi}^n(\Lambda x)$$

Then we recognize $\sum_x \Lambda^D$ as the discrete version of a volume integral, which is better and better approximated by a volume integral the smaller Λ is. When x occurs on lattice points, the argument of $\tilde{\phi}(\Lambda x)$ has arguments on a very fine grid, with apparent lattice constant Λ to the point where it is well approximated by a continuously varying function. The above monomial then becomes

$$\Lambda^{n\frac{D-2}{2}-D} \zeta^n \int d^D x \tilde{\phi}^n(x)$$

where the rescaled field now has cutoff of order one,

$$\tilde{\phi}(x) = \int_{p<1} \frac{d^D p}{(2\pi)^D} e^{ipx} \tilde{\phi}(p) \tag{38}$$

10. *Consider the quadratic terms in* $S[\phi^<]$,

$$\frac{1}{2}\sum_{x,y}\phi^<(x)\left(\Delta^{-1}(x,y)-\delta_{x,y}\right)\phi^<(y)$$

Find the form of these quadratic terms after the re-scaling

$$\phi(x) \;=\; \Lambda^{\frac{D-2}{2}}\zeta\tilde{\phi}(\Lambda x)$$

and in the limit where $\Lambda << 1$.

5.3 The effective action

Now, let us assume that we have done the two steps of the renormalization group process which we have described above. We are left with an effective action for $\phi_<(x)$ which resembles a continuum field theory. The effective action has the expansion in "local operators"

$$\mathbf{S}_{\text{eff}}[\phi] = \int dx\left\{\frac{1}{2}(\vec{\nabla}\phi(x))^2+\sum_{\mathcal{O}}\lambda_{\mathcal{O}}\mathcal{O}(x)\right\} \tag{39}$$

A local operator is defined as a monomial in the fields $\phi(x)$, all evaluated at the same point, and their derivatives, also evaluated at the same point. A generic example of a local operator is

$$\mathcal{O}(x) \;=\; \nabla_1^{k_1}\dots\nabla_d^{k_d}\phi(x)\nabla_1^{q_1}\dots\nabla_d^{q_d}\phi(x)\dots\nabla_1^{\ell_1}\dots\nabla_d^{\ell_d}\phi(x) \tag{40}$$

The local operators which appear in the effective action are governed by the symmetry of the lattice and the spin system. In our case, this means that, inside a given operator, each component of the gradient operator $\vec{\nabla}$ must appear an even number of times. The terms must also be invariant under the rotations which preserve the lattice, that is, rotations by $\frac{\pi}{2}$ radians about any coordinate axis. The lowest nonzero order in derivatives and operators which respects these symmetries is

$$\phi^2(x)$$

For operators which have two fields and two derivatives, the only possibility is

$$(\vec{\nabla}\phi(x))^2$$

In this latter case, we can see that the operator has an enhanced symmetry – a symmetry under rotations of the coordinates x which is not a symmetry of the theory at the level of the lattice. If for some reason this became the most important operator with derivatives, the spin system, when probed at wave-lengths which are much larger than the lattice spacing, would have an emergent symmetry under spatial rotations. We will see that this is indeed the case for the Ising model.

We shall always rescale the variabvle $\phi(x) \to$ constant $\phi(x)$, so that the term with the local operator $\frac{1}{2}(\vec{\nabla}\phi(x))^2$ has coefficient one in the effective action.

11. *Demonstrate that the integral of the local operator*

$$\int d^D x \frac{1}{2} \vec{\nabla}\phi(x) \cdot \vec{\nabla}\phi(x)$$

is left unchanged by the substitution $\phi(x) \to \Lambda^{\frac{D-2}{2}} \phi(\Lambda x)$.

12. *How is the integral of the local operator*

$$\int d^D x \ \phi^n(x)$$

transformed by the substitution of $\phi(x) \to \Lambda^{\frac{D-2}{2}} \phi(\Lambda x)$.

A useful characterization of local operators will be by their scaling dimensions. This is usually called the "classical dimension" or "engineering dimension" since it comes from simple dimensional analysis (and there is a more sophisticated definition of dimension). When the first term in the effective action is

$$\int dx \frac{1}{2} (\vec{\nabla}\phi(x))^2$$

we can use the fact that the action, in order to appear as the argument of a transcendental function such as the exponential, must be dimensionless to find that the field $\phi(x)$ must have a dimension given by

$$\dim(\phi) = \frac{D-2}{2}$$

which we take to mean that its the values of the field are measured in units which are the inverse of distance units to the power of $\frac{D}{2} - 1$.

Then, the local operator $\mathcal{O}(x)$, which is a product of ϕ's and their derivatives evaluated at the point x, has dimension

$$\dim(\mathcal{O}) = \left(\frac{D}{2} - 1\right)(\text{number of } \phi'\text{s}) + (\text{number of } \nabla'\text{s})$$

The lowest dimensional operator which has Z_2 stnnetrt is $\phi^2(x)$ which has

$$\dim\left(\frac{1}{2}\phi^2(x)\right) = D - 2$$

Also, some other examples are

$$\dim\left(\frac{1}{2}(\vec{\nabla}\phi(x))^2\right) = D$$

$$\dim\left(\frac{1}{4!}\phi^4(x)\right) = 2D - 4$$

$$\dim\left(\frac{1}{6!}\phi^6(x)\right) = 3D - 6$$

$$\dim\left(\frac{1}{2}\vec{\nabla}^2\phi(x)\vec{\nabla}^2\phi(x)\right) = D + 2$$

The above list of operators is the set of operators that have Z_2 symmetry, they are invariant under $\phi \to -\phi$. If we look at non Z_2 symmetric operators, there are more low dimensional ones such as $\phi(x)$, $\phi^3(x)$, $\phi^5(x)$. These are not candidates for the effective action of the Ising model since they do not respect the Z_2 symmetry.

Then, once we have a way of counting the dimensions of operators, there is a useful classification of local operators.
A local operator is

(a) $\mathcal{O}(x)$ is a "relevant operator" if $\dim[\mathcal{O}] < D$.

(b) $\mathcal{O}(x)$ is a "marginal operator" if $\dim[\mathcal{O}] = D$.

(c) $\mathcal{O}(x)$ is an "irrelevant operator" if $\dim[\mathcal{O}] > D$.

13. *Show that, under the scale transformation,* $\phi(x) \to \Lambda^{\frac{D-2}{2}}\phi(\Lambda x)$*, since* $\Lambda << 1$*, the coupling constants of relevant operators grow, of marginal operators remain unchanged and of irrelevant operators shrink.*

14. *Make a list of the relevant and marginal operators in* $D = 3, 4, 5, 6$. We can see that, if we assume that the irrelevant operators shrink away to having such small coupling constants that they can be ignored entirely, the effective actions in these dimensions simplify considerably. If the dimension is low enough, some non-quadratic behaviour survives.

6 Renormalization group functions

We have seen that, under the renormalization group transformation, all but a few of the operators in the effective action get small coefficients, and the rescaling can be done in such a way as to make them arbitrarily small. We are left with a simplified effective action

$$D > 4: \quad \mathbf{S}_{\text{eff}}[\phi] = \int d^D x \left(\frac{1}{2} \nabla\phi\nabla\phi + \frac{\tau}{2}\phi^2 \right) \tag{41}$$

$$3 < D < 4: \quad \mathbf{S}_{\text{eff}}[\phi] = \int d^D x \left(\frac{1}{2} \nabla\phi\nabla\phi + \frac{\tau}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right) \tag{42}$$

$$2 < D < 3: \quad \mathbf{S}_{\text{eff}}[\phi] = \int d^D x \left(\frac{1}{2} \nabla\phi\nabla\phi + \frac{\tau}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 \right) \tag{43}$$

The remainder of the problem at hand has simplified, particularly in $D > 4$ where the remaining integration of the ϕ-variables is Gaussian. This is the reason why the critical behaviour in $D > 4$ is described by mean field theory. However, in $D < 4$, we are left with a non-trivial problem. In $3 < D < 4$, the coefficient of the operator $\frac{\lambda}{4!}\phi^4$ grows as we lower the cutoff. The only way to control this growth is to remain close to four dimensions. In $4 - \epsilon$ dimensions, where ϵ is small, the growth is $\lambda \to \Lambda^{-\epsilon}\lambda$. It is possible, and it indeed happens that, this growth can be compensated by non-linear effects coming from the non-Gaussianity of the integrations that brought us to this point. In the following we will see how this can happen.

For this purpose, we will make an estimate of how τ and λ change when we do a further renormalization group transformation. To do this, we begin with a given effective action. We assume that λ, the constant which governs the accuracy of any computation that we can do, is small. Then we implement step 1 of the further renormalization group transformation by integrating over the ϕ's with wave-vectors between two cutoffs, $\Lambda < |p| < 1$ in order to produce another effective action. Then we implement step 2 of the renormalization group. We re-scale to set the cutoff back to 1. We are left with a certain Λ-dependence of the coupling constant which will help us understand how, beyond the simple rescaling that we have already discussed, the coupling constants depend on Λ.

As we have already emphasized, the accuracy of our computation relies on the assumption that λ is small. This justifies using the saddle point technique. In order for this assumption that λ is small to be consistent, we need to assume that we are close to four dimensions, that is $D = 4-\epsilon$ where ϵ is small. In fact, the appropriate size of ϵ will be $\epsilon \sim \lambda$ and our approximation of the integral that we have to do will be a double expansion in ϵ and λ, assuming that they are both small parameters of about the same magnitude. This double expansion is called the "epsilon expansion".

The decomposition to small and large wave-number modes is,

$$\phi(x) = \phi^<(x) + \phi^>(x) \tag{44}$$

$$\phi^<(x) = \int_{|\vec{k}<\Lambda} \frac{d\vec{k}}{(2\pi)^{D/2}} e^{i\vec{k}x}\phi(\vec{k}) \ , \quad \phi^>(x) \quad = \int_{\Lambda\leq|\vec{k}|<1} \frac{d\vec{k}}{(2\pi)^{D/2}} e^{i\vec{k}x}\phi(\vec{k}) \tag{45}$$

and the integration measure factorizes, $\int[d\phi] = \int[d\phi^<]\int[d\phi^>]$. We define a new effective action by

$$e^{-\hat{\mathbf{S}}_{\rm eff}[\phi^<]} = \int [d\phi^>] e^{-\mathbf{S}_{\rm eff}[\phi^< + \phi^>]} \tag{46}$$

The effective action in dimensions less than but in the vicinity of $D = 4$ is given by

$$\mathbf{S}_{\rm eff}[\phi^< + \phi^>] = \int d^{4-\epsilon}x \left\{ \frac{1}{2}(\vec{\nabla}\phi^<)^2 + \frac{\tau}{2}(\phi^<)^2 + \frac{\lambda}{4!}(\phi^<)^4 \right\} \tag{47}$$

$$+ \int d^{4-\epsilon}x \phi^> \left\{ -\vec{\nabla}^2\phi^< + \tau\phi^< + \frac{\lambda}{3!}(\phi^<)^3 \right\} \tag{48}$$

$$+ \int d^{4-\epsilon}x \frac{1}{2}\phi^> \left[-\vec{\nabla}^2 + \tau + \frac{1}{2}\lambda(\phi^<)^2 \right] \phi^> + \ldots \tag{49}$$

We will drop the terms in the second line. The quadratic terms there vanish because the fields would have to have the same wave-number and that is impossible. The term with $\int d^{4-\epsilon}x\phi^{>}\frac{\lambda}{3!}(\phi^{<})^3$ could be nonzero when the three wave-vectors in the $\phi^{<}$, which each have magnitudes less that Λ, add up to a wave-vector larger than Λ and it could at most contribute to a ϕ^6 term which we could take into account if we need it. Then we do the Gaussian integral over $\phi^{>}$.

15. *Show that we get a new effective action which depends on $\phi^{<}$ – we will drop the super-script – and has the form*

$$\mathbf{S}_{\rm eff}[\phi] = \int d^{4-\epsilon}x\left\{\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\tau}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right\} + \frac{1}{2}\mathrm{Tr}\ln\left[-\vec{\nabla}^2 + \tau + \frac{1}{2}\lambda\phi^2\right] + \ldots \tag{50}$$

We can find the corrections to the coefficients of all of the local operators which are simple monomials $\int d^{4-\epsilon}x\phi^n(x)$ in the effective action by evaluating the trace log term with ϕ^2 set equal to a constant. Then we could also look at the shift in local operators which contain derivatives. We will not do this, although we note that it is known that the $(\nabla\phi)^2$ term does not get a correction from the trace-logarithm term in (50).

16. *Show that, if we assume that ϕ^2 appearing inside the logarithm in equation (50) is a constant, the trace logarithm becomes*

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\ln\left[-\vec{\nabla}^2 + \tau + \lambda\phi^2\right] &= V\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\ln(p^2+\tau) \\ &+ \int d^4x\sum_{n=1}^{\infty}\frac{1}{2}\frac{(-1)^{n-1}}{n}\left(\frac{\lambda\phi^2}{2}\right)^n\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(\tau+p^2)^n}\end{aligned} \tag{51}$$

and, up to a ϕ-independent constant, the effective action is

$$\begin{aligned}\mathbf{S}_{\rm eff}[\phi] &= \int d^{4-\epsilon}x\left\{\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\tau}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right\} \\ &+ \sum_{n=1}^{\infty}\left[(-1)^{n-1}\frac{(2n)!\lambda^n}{2^n n}\frac{1}{16\pi^2}\int_{\Lambda}^{1}\frac{p^3dp}{(\tau+p^2)^n}\right]\int d^4x\frac{\phi^{2n}(x)}{(2n)!} \\ &+ \text{irrelevant operators with derivatives of } \phi\end{aligned} \tag{52}$$

where the cutoff is now Λ.

Then we can implement the second step of the renormalization group transformation.

17. *Show that, if we rescale – with the substitution*

$$\phi(x) \;\rightarrow\; \Lambda^{1-\frac{\epsilon}{2}}\phi(\Lambda x)$$

we obtain a new effective action, for a theory with cutoff equal to one, which is

$$\begin{aligned}
\mathbf{S}_{\text{eff}}[\phi] = \int d^{4-\epsilon}x\left\{\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{\Lambda^2}\frac{\tau}{2}\phi^2 + \Lambda^{-\epsilon}\frac{\lambda}{4!}\phi^4\right\} \\
+ \sum_{n=1}^{\infty}\left[(-1)^{n-1}\frac{(2n)!\lambda^n}{2^n n}\frac{\Lambda^{2n-4}}{16\pi^2}\int_{\Lambda}^{1}\frac{p^3dp}{(\tau+p^2)^n}\right]\int d^4x\frac{\phi^{2n}(x)}{(2n)!} \\
+ \text{irrelevant operators with derivatives of } \phi
\end{aligned} \tag{53}$$

Now we see that the two coupling constants τ and λ that were in our effective action have been changed, as

$$\tau(\Lambda) = \frac{1}{\Lambda^2}\tau(1) + \frac{1}{\Lambda^2}\frac{2\lambda}{2}\frac{1}{16\pi^2}\int_{\Lambda}^{1}\frac{p^3dp}{(\tau+p^2)}$$

$$\lambda(\Lambda) = \Lambda^{-\epsilon}\lambda - \frac{3\lambda^2}{16\pi^2}\int_{\Lambda}^{1}\frac{p^3dp}{(\tau+p^2)^2}$$

and the coefficient of ϕ^n with $n > 4$ is $\sim \Lambda^{2n-4}$ for small Λ, which goes to zero as $\Lambda \to 0$. We will ignore these terms.

18. *Show that the beta functions for the coupling constants are given by the expresions*

$$\beta_\tau(\Lambda) = \Lambda\frac{\partial}{\partial\Lambda}\tau(\Lambda) = -2\tau(\Lambda) - \frac{\lambda}{16\pi^2}\frac{\Lambda^2}{\tau+\Lambda^2} + \ldots \tag{54}$$

$$\beta_\lambda(\Lambda) = \Lambda\frac{\partial}{\partial\Lambda}\lambda(\Lambda) = -\epsilon\lambda(\Lambda) + \frac{3\lambda^2}{16\pi^2}\frac{\Lambda^4}{(\tau+\Lambda^2)^2} + \ldots \tag{55}$$

Using the beta functions we can form a flow equation which governs the evolution of the coupling constants τ and λ as the cutoff Λ is lowered,

$$t = \ln\frac{1}{\Lambda}$$

$$\frac{d}{dt}\tau(t) = -\beta_\tau(\tau(t), \lambda(t), \Lambda = e^{-t}) \tag{56}$$

$$\frac{d}{dt}\lambda(t) = -\beta_\lambda(\tau(t), \lambda(t), \Lambda = e^{-t}) \tag{57}$$

The flow starts at some initial values $(\tau(0), \lambda(0))$ and follows a "renormalization group trajectory" that is governed by equations (56) and (57).

Important information is contained in the zeros of the set of beta functions. These are points where the flow stops. They are therefore called "fixed points".

19. *Show that the beta functions have two joint zeros,*

$$\tau_0^* = 0 \ , \quad \lambda_0^* = 0 \tag{58}$$

and

$$\tau_{WF}^* = -\frac{1}{6}\epsilon \ , \quad \lambda_{WF}^* = \frac{16\pi^2}{3}\epsilon \tag{59}$$

where we have approximated the right-hand-sides by keeping only the leading order term in ϵ. Note that the fixed points are Λ-independent. One of them is at the origin in the space of coupling constants. This is sometimes called a "Gaussian fixed point". The other is at a point in the $\lambda-\tau$-plane away from the origin and it is called the "Wilson-Fisher fixed point".

To understand the significance of the Gaussian fixed point, we can linearize the flow equation near it

$$\tau \approx \tau_0^* \ , \quad \lambda \approx \lambda_0^* \quad , \quad t = \ln(1/\Lambda) \tag{60}$$

$$\frac{d}{dt}\begin{bmatrix}\tau(t) \\ \lambda(t)\end{bmatrix} \approx \begin{bmatrix}2 & \frac{1}{16\pi^2} \\ 0 & \epsilon\end{bmatrix}\begin{bmatrix}\tau(t) \\ \lambda(t)\end{bmatrix} + \dots \tag{61}$$

The matrix $\begin{bmatrix}2 & \frac{1}{16\pi^2} \\ 0 & \epsilon\end{bmatrix}$ has an eigenvalue 2 with eigenvector $\begin{bmatrix}1 \\ 0\end{bmatrix}$ and an eigenvalue ϵ with eigenvector $\frac{1}{\sqrt{(16\pi^2)^2+1/(2-\epsilon)^2}}\begin{bmatrix}-1/(2-\epsilon) \\ 16\pi^2\end{bmatrix}$.

20. *Show that the solution of equations (56) and (57) in the vicinity of the Gaussian fixed point is*

$$\begin{bmatrix}\tau(t) \\ \lambda(t)\end{bmatrix} = \begin{bmatrix}\tau(0) + \frac{\lambda(0)}{16\pi^2(2-\epsilon)} \\ 0\end{bmatrix} e^{2t} + \begin{bmatrix}-\frac{\lambda(0)}{16\pi^2(2-\epsilon)} \\ \lambda(0)\end{bmatrix} e^{\epsilon t} + \dots$$

From this solution we see that the flow, as Λ is lowered, is always away from the Gaussian fixed point $(\tau_0^*, \lambda^*) = (0, 0)$.

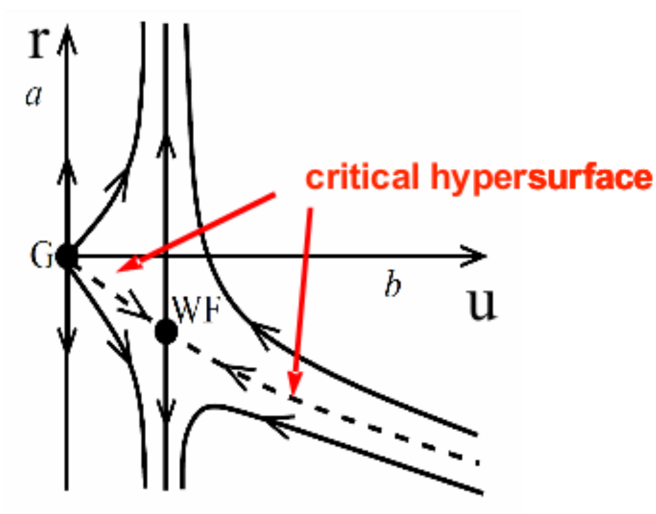

Figure 1: The flow diagram corresponding to the beta functions for τ and λ is depicted. Values of τ are plotted on the vertical axis and λ on the horizontal axis. The trivial (Gaussian) fixed point (τ_0^*, λ_0^*) is located at the point labeled G. The directions of the flow are indicated by arrows. The "critical hypersurface" is the location of values of τ and λ from which a flow will terminate at the Wilson-Fisher fixed point, denoted WF.

21. *Find the stability matrix,* $\left.\frac{\partial\beta_i}{\partial\lambda_j}\right|_{\lambda^*}$ *at the Wilson-Fisher fixed point and write down the linearized flow equation for the immediate vicinity of the Wilson-Fisher fixed point. Show that the stability matrix has one positive and one negative eigenvalue. Find an explicit solution of the flow equations in the vicinity of the Wilson-Fisher fixed point.*

The nature of the coupling constant flow is sketched in figure 1. In that figure the dotted line, termed stability hypersurface, is the subspace of coupling constant space along which the initial condition of the flow can be placed so that the flow takes it to the Wilson-Fisher fixed point. Placing the initial condition on the stability hypersurface is equivalent to tuning the temperature to the critical temperature. Flowing along the stability hypersurface is then equivalent to performing renormalization group transformations which lower the cutoff. This process eventually stabilizes when the flow reaches the Wilson-Fisher fixed point.

We note that the basin of attraction of the fixed point goes on to the region where λ is large, eventually out of the regime where our computation of the beta functions is legitimate. This means that if λ is larger, after some renormalization group transformations, it might still become small enough that the remainder of our analysis is correct. This is the miracle of the Wilson-Fisher fixed point – that it may have – and there is much evidence that it indeed does have validity beyond our simple computation.

7 Critical Exponents

To use the renormalization group we can make the following observation, the original problem for computing the partition function did not involve the final cutoff Λ. In fact, this final cutoff only appeared in our computations in the previous few pages, before that it was nowhere to be seen. That means, if we go back to the definition of a correlation function in the original model, it cannot depend on Λ, that is

$$\Lambda\frac{\partial}{\partial\Lambda} < \phi(x)\phi(y) >= 0$$

Once we have applied the renormalization group and introduced Λ, if we compute the correlation function in our effective theory, Λ will appear everywhere. We can make use of this fact. First of all, to get the effective theory, we have re-scaled $\phi(x)$ by Λ-dependent factors. So the original correlation function would be more like the expression

$$< \phi(x)\phi(y) >_{original} = Z(\Lambda) < \phi(x)\phi(y) >_{\text{effective}}$$

We could define another renormalization group function

$$\gamma(\tau, \lambda, \Lambda) = \frac{1}{2}\Lambda \frac{\partial}{\partial \Lambda} \ln Z$$

Then, the Λ-independence of the original correlation function could be written as an equation for the effective correlation function

$$0 = \left(\Lambda \frac{\partial}{\partial \Lambda} + 2\gamma(\tau, \lambda, \Lambda) + \beta_\tau(\tau, \lambda, \Lambda) \frac{\partial}{\partial \tau} + \beta_\lambda(\tau, \lambda, \Lambda) \frac{\partial}{\partial \lambda} \right) < \phi(x)\phi(y) >_{\text{effective}} \tag{62}$$

This is the renormalization group equation for a correlation function. It is sometimes called the Callan-Symanzik equation. If we sit on the fixed point, this equation tells us that as $\Lambda \to 0$,

$$0 = \left(\Lambda \frac{\partial}{\partial \Lambda} + 2\gamma(\tau^*_{WF}, \lambda^*_{WF}, \Lambda \to 0) \right) < \phi(x)\phi(y) >_{\text{effective}} \tag{63}$$

Emergent translation and rotation invariance tells us that $< \phi(x)\phi(y) >_{\text{effective}}$ is a function of $|x-y|$ only and dimensional analysis tells us that

$$< \phi(x)\phi(y) >_{\text{effective}} = \frac{f(|x-y|\Lambda)}{|x-y|^{D-2}} \tag{64}$$

Then, requiring it to obey equation (63) tells us that it must be of the form

$$< \phi(x)\phi(y) >_{\text{effective}} = \frac{[(|x-y|\Lambda)]^{2\gamma(\tau^*_{WF}, \lambda^*_{WF}, 0)}}{|x-y|^{D-2}} \sim \frac{1}{|x-y|^{D-2+2\gamma(\tau^*_{WF}, \lambda^*_{WF}, 0)}} \tag{65}$$

and we deduce the critical exponent

$$\eta = 2\gamma(\tau^*_{WF}, \lambda^*_{WF}, 0) \tag{66}$$

The renormalization group function $\gamma(\tau, \lambda, \Lambda)$ does not get contributions at order ϵ. Its first contribution is at order ϵ^2 which we have not computed yet. This means that, to leading order in the ϵ expansion, $\eta = 0$ and the dimension of the local operator $\phi(x)$ is $(D-2)/2$.

In order to compute the other critical exponents efficiently, it is extremely convenient to make an assumption. This assumption is called the hyperscaling hypothesis. It begins with the fact that, if we consider the Ising model at its critical point, $T = T_C$, and in the limit where we have lowered the cutoff Λ until the coupling constants have flown sufficiently close to their fixed points, for all intents and purposes the remaining effective model exhibits exact emergent scale invariance. If we move slightly away from the critical point, by moving the reduced temperature $t \equiv \frac{T-T_C}{T_C}$ away from zero, the correlation length becomes nonzero. In fact, how it depends on the reduced temperature defines one of the critical exponents,

$$\xi \sim t^{-\nu} \tag{67}$$

where we call $t \equiv \frac{T-T_C}{T_C}$ the reduced temperature. According to the scaling hypothesis, $\nu' = \nu$.

Hyperscaling assumes that, if we move off of the critical point by a slight amount, by making τ small but nonzero, there is only one relevant dimensionful parameter, the correlation length ξ, and and the behaviour near $t \sim 0$ of every other dimensionful quantity can be gotten from the correlation length by dimensional analysis.

We emphasize that hyperscaling is only a hypothesis. It could be right or wrong and it might not apply to every phase transition in every system. It does, however, fit what is known about the Ising model phase transition and many other phase transitions very well. So, let us make the hyper-scaling hypothesis.

Let us assume that the only dimensionful parameter is ξ. It is a length and it should be measured in units of length. It is related to the reduced temperature by equation (67).

The free energy is given by $F[T,V] = Vf(T)$ where $f(T)$ is the free energy density. The free energy density must have units of (distance)$^{-D}$ and hyperscaling then tells us that it must be of the form

$$f \sim \xi^{-D} \sim t^{\nu D}$$

From this equation, we get the specific heat critical exponent

$$c \sim \frac{\partial^2}{\partial t^2} f \sim t^{\nu D-2} \sim t^{-\alpha}$$

so we identify

$$\boxed{\alpha = 2 - \nu D} \tag{68}$$

Sometimes (68) is called the hyperscaling relation and the formulae which follow, that really do simply result from dimensional analysis, are called scaling relations.

The magnetization of the ordered phase is

$$m =< \phi >= t^{\beta}$$

which is dimensionally consistent only of the operator dimension of ϕ is equal to the dimension of t^{β} which means that

$$\Delta_{\phi} = \beta/\nu$$

where we recall that we named the scaling dimension of a local operator $\mathcal{O}(x)$ by $\Delta_{\mathcal{O}}$. From equation (65) we would say that the scaling dimension of ϕ is

$$\Delta_{\phi} = \frac{1}{2}(D - 2 + \eta)$$

so the scaling relation gives us the critcal exponent

$$\boxed{\beta = \frac{1}{2}(D - 2 + \eta)\nu}$$

We could re-introduce the B-term by adding

$$B \int d^D x \phi(x)$$

to the effective action. This is due to the fact that $\int d^D x \phi(x)$ contains only the zero wavenumber component of $\phi(x)$ which has not been integrated out so far, even though it has been rescaled. That rescaling gives it a dimension Δ_{ϕ} and the fact that the effective action must be dimensionless tells us that

$$\Delta_B + \Delta_{\phi} = D$$

or

$$\Delta_B = D - \frac{1}{2}(D - 2 + \eta) = \frac{1}{2}(D + 2 - \eta)$$

Then the definition of the magnetic susceptibility and dimensional analysis tells us that

$$\chi \sim \frac{\partial m}{\partial B} \sim t^{-\gamma} \sim (t^{-\nu})^{\Delta_\phi - \Delta_B}$$

which implies

$$\boxed{\gamma = \nu(2 - \eta)}$$

Finally when $t = 0$, and $B \sim 0$,

$$m \sim B^{1/\delta}$$

which, together with dimensional analysis, tells us that

$$\delta = \frac{\Delta_B}{\Delta_\phi}$$

or

$$\boxed{\delta = \frac{D + 2 - \eta}{D - 2 + \eta}}$$

If we gather the boxed equations above, we see that the critical exponents are determined if we know ν and η. We deduced η from the correlation function from which we also deduced the dimension of the local operator $\phi(x)$. It is

$$\frac{1}{2}(D - 2 + \eta) = \Delta_\phi$$

or

$$\boxed{\eta = 2\Delta_\phi - D + 2}$$

We already know that η vanishes at the first order in the ϵ expansion. The exponent ν can be gotten from the fact that the term $\int d^D x \tau \phi^2$ must be dimensioneless plus the assumption that $\tau \sim t$ so that $t \sim \xi^{\Delta_{\phi^2} - D}$.

$$\boxed{\nu = \frac{1}{D - \Delta_{\phi^2}}}$$

We have seen that important data for the determination of the critical exponents are the dimensions of some of the lower operators. If we define

$$\Delta_\phi = \text{ dimension of } \phi \tag{69}$$
$$\Delta_{\phi^2} = \text{ dimension of } \phi^2 \tag{70}$$
$$\Delta_{\phi^4} = \text{ dimension of } \phi^4 \tag{71}$$

then

$$\begin{array}{cccc} & d=2 & d=3 & d=4 \\ \Delta_\phi & \frac{1}{8} & 0.5181489(10) & 1 \\ \Delta_{\phi^2} & 1 & 1.412625(10) & 2 \\ \Delta_{\phi^4} & 4 & 3.82966(9) & 4 \end{array} \tag{72}$$

and the critical exponents are

$$\begin{array}{ccccc} & D=2 & D=3 & D=4 & \text{scaling} \\ \alpha & 0 & 0.11008(1) & 0 & 2-\frac{D}{D-\Delta_{\phi^2}} \\ \beta & \frac{1}{8} & 0.326419(3) & \frac{1}{2} & \frac{\Delta_\phi}{D-\Delta_{\phi^2}} \\ \gamma & \frac{7}{4} & 1.237075(10) & 1 & \frac{D-2\Delta_\phi}{D-\Delta_{\phi^2}} \\ \delta & 15 & 4.78984(1) & 3 & \frac{D-\Delta_\phi}{\Delta_\phi} \\ \eta & \frac{1}{4} & 0.036298(2) & 0 & 2\Delta_\phi-D+2 \\ \nu & 1 & 0.629971(4) & \frac{1}{2} & \frac{1}{D-\Delta_{\phi^2}} \end{array} \tag{73}$$

The $D=2$ exponents quoted above are found from Onsager's exact solution of the 2D Ising model. The $D=3$ exponents quoted above are results of the conformal bootstrap, which is currently the approach which offers the most accuracy. The $D=4$ exponents quoted are those of mean field theory.

The critical exponents for for the $D=4-\epsilon$ dimensional Ising model in the epsilon expansion are also easily computed from the scaling relations. The results are

$$\alpha=\frac{\epsilon}{6}, \gamma=1+\frac{\epsilon}{6},\ \beta=\frac{1}{2}-\frac{\epsilon}{6},\ \delta=3+\epsilon,\ \eta=\frac{\epsilon^2}{54},\ \nu=\frac{1}{2}+\frac{\epsilon}{12} \tag{74}$$

One can compare these low order and still rather crude leading order computations (at $\epsilon\to 1$) with the known exponents of the D=3 Ising model given in the table above and you can see they are already a promising fit.