Tutorial 3: The $O(\mathcal{N})$ Model

The Ising model is a simple model of a magnetic system which has spin variables $\sigma_{\vec{n}}$ residing on a hyper-cubic lattice whose sites are labeled by $\vec{n}$. Each of the spins can take the values +1 or −1. The energy of a configuration of the spins is given by the Hamiltonian

$$H[\sigma] \;=\; -J\sum_{\vec{n}}\sum_{\mu}\sigma_{\vec{n}}\sigma_{\vec{n}+\vec{\mu}} - B\sum_{\vec{n}}\sigma_{\vec{n}}$$

The vectors $\vec{\mu}$ in the summation run over the positive unit vectors oriented positively along the d Cartesian coordinate axes in d-dimensional Euclidean space. They link the spins to their nearest neighbours. Confining the $\vec{\mu}$'s to positive unit vectors counts each link connecting nearest neighbours precisely once.

We have and will continue to confine our attention to the example of a hyper-cubic lattice. The sites of the lattice are located at the points

$$\vec{x} = a\vec{n} = a\,(n_1, n_2, ..., n_d)$$

Here, a is the lattice constant, or lattice spacing, which is the distance between nearest neighbour sites on the lattice. The entries $n_1, n_2, ..., n_d$ in the vector in the formula above are integers which take the values $n_j = 0, 1, 2, ..., N-1$ and we use periodic boundary conditions which identifies $n_j \sim n_j + N$. The total number of lattice sites is then N^d and the volume of the lattice is

$$\mathcal{V} = N^d a^d$$

The partition function of the Ising model is

$$Z[T, N, B] = \sum_{\text{spins}} e^{-\frac{1}{k_B T}H[\sigma]} = \sum_{\text{spins}} e^{\frac{J}{k_B T}\sum_{\vec{n},\vec{\mu}}\sigma_{\vec{n}}\sigma_{\vec{n}+\vec{\mu}} + \frac{B}{k_B T}\sum_{\vec{n}}\sigma_{\vec{n}}}$$

where the sum over spins is the sum over the two possible spin states of the spins which are located at each site of the lattice.

In lectures and in the lecture notes, we showed that a Gaussian transform could be used to write the Ising model partition function in the form of a ratio of integrals which resembles a lattice field theory

$$Z[T,N,B] = \frac{\left[\prod_{\vec{n}} \int_{-\infty}^{\infty} d\phi_{\vec{n}}\right] \exp\left(-\frac{1}{2}\sum_{\vec{n},\vec{n}'} \phi_{\vec{n}} \Delta^{-1}(n,n')\phi_{n'} + \sum_{\vec{n}} \ln 2\cosh\left(\phi_{\vec{n}} + \frac{B_{\vec{n}}}{k_B T}\right) - \frac{cN^d}{2k_B T}\right)}{\left[\prod_{\vec{n}} \int_{-\infty}^{\infty} d\tilde{\phi}_{\vec{n}}\right] \exp\left(-\frac{1}{2}\sum_{\vec{n},\vec{n}'} \tilde{\phi}_{\vec{n}} \Delta^{-1}(n,n')\tilde{\phi}_{n'}\right)} \tag{1}$$

where the "lattice field" is $\phi_{\vec{n}}$. The real number $\phi_{\vec{n}}$ is the magnitude of this lattice field at the lattice site that is labeled by $\vec{n}$, or located at the position $\vec{x} = a\vec{n}$. The function $\Delta^{-1}(n,n')$ is defined by its Fourier transform,

$$\Delta^{-1}(n,n') = \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot(\vec{n}-\vec{n}')}}{N^d} \frac{k_B T}{2J\sum_{j=1}^{d} \cos p_j + c}$$

Here, the quantity $\vec{p}$ which is summed over in the Fourier transform is a vector that takes on values

$$\vec{p} = \frac{2\pi}{N}\left(\ell_1, \ell_1, \ldots, \ell_d\right)$$

with $\ell_1, \ell_2, \ldots, \ell_d$ each being integers in the range $\ell_j = 01, 2, \ldots, N-1$ and, these also have periodic boundary conditions so that ℓ_j is identified with $\ell_j + N$. Notice that there are exactly the same number of distinct values of $\vec{p}$ as there are distinct values of $\vec{n}$, N^d in both cases.

The lattice Fourier transform and the inverse Fourier transform of the field are

$$\phi_{\vec{p}} = \sum_{\vec{n}} \frac{e^{-i\vec{p}\cdot\vec{n}}}{N^{d/2}} \phi_{\vec{n}} \tag{2}$$

and

$$\phi_{\vec{n}} = \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{n}}}{N^{d/2}} \phi_{\vec{p}} \tag{3}$$

respectively. We denote the Fourier transform of $\phi_{\vec{n}}$ by $\phi_{\vec{p}}$. We use the same symbol for these two different functions and distinguish them by their argument – if the argument is a lattice position (or label of a position in this case), it is the lattice field. If the argument is the lattice wave-vector, it is the

Fourier transform of the lattice field. The normalization of the transforms in equations (2) and (3) is fixed so that, if we consider $M_{\vec{x}\vec{p}} = \frac{e^{i\vec{p}\cdot\vec{n}}}{N^{d/2}}$ as a matrix, it is a unitary matrix.

A sometimes clearer notation, which we will use extensively, denotes the lattice field as a function of the position argument $\phi(\vec{x})$ where $x = a\vec{n}$. We will take this to mean exactly the same thing as $\vec{\phi}_{\vec{n}}$, that is

$$\phi(\vec{x}) \ \equiv \ \phi_{\vec{n}} \ , \quad \vec{x} = a\vec{n}$$

We can also write the Fourier transform and the inverse Fourier transform in this notation,

$$\phi(\vec{k}) = \sum_{\vec{x}} \frac{e^{-i\vec{k}\cdot\vec{x}}}{N^{d/2}} \phi(\vec{x})$$

and

$$\phi(\vec{x}) = \sum_{\vec{k}} \frac{e^{i\vec{k}\cdot\vec{x}}}{N^{d/2}} \phi(\vec{k})$$

Here, $\vec{k}$ is the wave-vector, $\vec{k} \equiv \frac{1}{a}\vec{p}$, defined as the number of wave-lengths per unit distance. the position, $\vec{x}$, has the dimensions of distance, and the wave-vector, $\vec{k}$, has dimensions of inverse distance.

We will also be interested in the limit where N goes to infinity, where the lattice is infinitely large. In that case, the wave-vector $\vec{k} = (k_1, k_2, ..., k_d)$ changes from the discrete variable that we have been discussing above to a continuous variable whose components occupy the interval

$$-\frac{\pi}{a} < k_j \leq \frac{\pi}{a}$$

with the periodic identification

$$k_j \ \sim \ k_j + \frac{2\pi}{a}$$

The space of all distinct values of $\vec{k}$ is a d-dimensional hyper-torus. It is called the "Brillouin zone" and we sometimes label it as Ω_B.

The summations over discrete wave-vectors which occurred in the inverse Fourier transforms that we discussed above are replaced by volume integrals over the Brillouin zone by the prescription

$$\sum_{\vec{k}} \ \to \ N^d a^d \int_{\Omega_B} \frac{d\vec{k}}{(2\pi)^d} \ = \ \mathcal{V} \int_{\Omega_B} \frac{d\vec{k}}{(2\pi)^d}$$

Here, we have used an abbreviated notation

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_2}{2\pi} \cdots \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_d}{2\pi} ... \equiv \int_{\Omega_B} \frac{d\vec{k}}{(2\pi)^d} ...$$

The orthogonality and completeness relations for plane waves become

$$a^d \int_{\Omega_B} \frac{d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}} \ e^{-i\vec{k}\cdot\vec{y}} = \delta_{\vec{x},\vec{y}}$$

$$\sum_{\vec{x}} e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}} = \frac{(2\pi)^d}{a^d} \delta(\vec{k} - \vec{k}')$$

where the Kronecker delta is defined as

$$\delta_{\vec{x},\vec{y}} \ \equiv \ \begin{cases} 1 & \text{if } \vec{x} = \vec{y} \\ 0 & \text{if } \vec{x} \neq \vec{y} \end{cases}$$

and in the last formula, $\delta(\vec{k} - k')$ is the d-dimensional Dirac delta function with the property that

$$\int_{\Omega_B} d\vec{k}' \ \delta(\vec{k} - \vec{k}') f(\vec{k}') = f(\vec{k})$$

for any square-integrable function $f(\vec{k})$.

In infinite volume, the Fourier transforms of the lattice fields is

$$\phi(\vec{x}) = \int_{\Omega_B} \frac{d\vec{k}}{(2\pi/a)^{d/2}} e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k})$$

$$\phi(\vec{k}) = \sum_{\vec{x}} \frac{1}{(2\pi/a)^{d/2}} \ e^{-i\vec{k}\cdot\vec{x}} \ \phi(\vec{x})$$

and the expressions for $\Delta(x, y)$ and $\Delta^{-1}(x, y)$ are

$$\Delta(x, y) = a^d \int_{\Omega_B} \frac{d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{2J \sum_{j=1}^d \cos ak_j + c}{k_B T}$$

$$\Delta^{-1}(x, y) = a^d \int_{\Omega_B} \frac{d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{k_B T}{2J \sum_{j=1}^d \cos ak_j + c}$$

respectively.

1. *Use the formulae above to confirm that*

$$\sum_{\vec{y}} \Delta^{-1}(x,y)\Delta(y,z) \;=\; \delta_{\vec{x},\vec{z}}$$

We can use the above considerations to change our notation for the partition function to the following one, which we have used as a starting point for computations [1]

$$Z[T,N,B] = \frac{1}{(2\pi)^{N^d/2}\sqrt{\det\Delta}} \int [d\phi(\vec{x})] \exp\left(-S[\phi]\right) \tag{4}$$

$$S[\phi] = S_0[\phi] + V[\phi] \tag{5}$$

$$S_0[\phi] = \frac{1}{2}\sum_{\vec{x},\vec{y}} \phi(x)\Delta^{-1}(x,y)\phi(\vec{y}) \tag{6}$$

$$V[\phi] = \sum_{\vec{x}} V\left(\phi(\vec{x})\right) \tag{7}$$

$$V(\phi(\vec{x})) = -\ln 2\cosh\left(\phi(\vec{x}) + \frac{B}{k_BT}\right) + \frac{c}{2k_BT} \tag{8}$$

Through the construction that leads to the above equations for the partition function, we can find the correlation functions of the Ising model spins.The spin and lattice field expectation value and correlation functions are related by

$$m(\vec{x}) \equiv < \sigma(\vec{x}) > = \sum_{\vec{y}} \Delta^{-1}(x,y) < \phi(\vec{y}) > \tag{9}$$

$$\chi(x,y) \equiv < \sigma(\vec{x})\sigma(\vec{y}) >_C = \sum_{\vec{w}\vec{z}} \Delta^{-1}(x,w)\left\langle\ \phi(\vec{w})\phi(\vec{z})\ \right\rangle_C \Delta^{-1}(z,y) - \Delta^{-1}(x,y) \tag{10}$$

where the subscript $_C$ denotes "connected correlation function". $m(\vec{x})$ is the magnetization density and $\chi(x,y)$ is the magnetic susceptibility.

[1] Recall that the determinant is defined as the product over the eigenvalues so that

$$\det\Delta(n,n') = \prod_{\vec{p}} \Delta(p) = \exp\left(\sum_{p} \ln\Delta(\vec{p})\right) = \exp\left(\sum_{\vec{p}} \ln\left[\frac{c}{k_BT} + \sum_{j=1}^{d} \frac{2J}{k_BT}\cos p_j\right]\right)$$

$$= \exp\left(V\int_{\Omega_B} \frac{d\vec{p}}{(2\pi)^d} \ln\left[\frac{c}{k_BT} + \sum_{j=1}^{d} \frac{2J}{k_BT}\cos p_j\right]\right)$$

where $V = N^d a^d$ is the (infinite) volume.

We note that equation (4) is still an exact expression for the partition function of the Ising model in d dimensions. We are not able to do the integrations that would be necessary to evaluate it and we have to resort to techniques which address the problem in a less direct fashion and, of course, provide a less fulsome answer. What we hope to achieve is an understanding of the non-analytic behaviour of the partition function and the correlation functions as the external parameters, T and B are tuned to the critical point, that is the point (T_c, B_c) in the T-B-plane where the Ising model undergoes a second order phase transition. We know by Z_2 symmetry that $B_c = 0$.

In the following we will observe that we could do a similar analysis of other classical models of magnetic systems. For example, if rather than the spins taking only the two discrete values $\sigma_{\vec{n}} = \pm 1$, as they do for the Ising model, we could define a system where, for each lattice site $\vec{n}$, the spin $\vec{\sigma}_{\vec{n}}$ is an $\mathcal{N}$-component variable,

$$\vec{\sigma}_{\vec{n}} \equiv \left(\sigma^1_{\vec{n}}, \sigma^2_{\vec{n}}, ..., \sigma^{\mathcal{N}}_{\vec{n}}\right) \tag{11}$$

with the property that, for each lattice site labeled by $\vec{n}$, the spin $\vec{\sigma}_{\vec{n}}$ is a unit vector, that is,

$$\vec{\sigma}_{\vec{n}} \cdot \vec{\sigma}_{\vec{n}} = (\sigma^1_{\vec{n}})^2 + (\sigma^2_{\vec{n}})^2 + ... + (\sigma^{\mathcal{N}}_{\vec{n}})^2 = 1 \tag{12}$$

With the hope that this does not cause confusion, we will use the overhead arrow to denote both the $\mathcal{N}$-vector $\vec{\sigma}$ and the d-vector $\vec{n}$. (This is standard in literature.) The space of possible orientations of this unit vector is equivalent the $(\mathcal{N}-1)$-dimensional unit sphere, denoted $S^{(\mathcal{N}-1)}$, which is embedded in $\mathcal{N}$-dimensional Euclidean space by the constraint in equation (12) on its coordinates.

In order to make a physical model where the spins are the degrees of freedom, we need to write down a Hamiltonian. We take the Hamiltonian to be similar to that of the classical Ising model,

$$H = -J \sum_{\vec{n},\mu} \vec{\sigma}_{\vec{n}} \cdot \vec{\sigma}_{\vec{n}+\mu} - \sum_{\vec{n}} \vec{B} \cdot \vec{\sigma}_{\vec{n}} \tag{13}$$

This Hamiltonian and the prescription for finding the partition function which is given in equation (14) below define a classical statistical model of this spin system.

The Ising model is recovered as a special case of the model that we are considering by putting $\mathcal{N} = 1$. The case with $\mathcal{N} = 2$ is called the "classical X-Y model" and with $\mathcal{N} = 3$ it is called the "classical Heisenberg model". The case with general $\mathcal{N}$ is called the "$O(\mathcal{N})$ model". The name comes from the symmetry of the model when $\vec{B} = 0$, which is symmetry under rotations in $\mathcal{N}$-dimensional space. The set of all such rotations comprise the Lie group $O(\mathcal{N})$. (Beyond this name, we are not assuming any knowledge of Lie group theory.)

As in the case of the Ising model, the constant J governs how strongly the neighbouring spins are coupled to each other. The quantity $\vec{B}$ is an N-component object which plays the role of an external magnetic field. The signs of the terms in the Hamiltonian, if J is a positive number, are arranged so that so that the minimum energy state is Ferromagnetic, that is, it has all of the spins aligned in the same direction in the $\mathcal{N}$-dimensional space that they live in, and that direction is the direction of the external field, $\vec{B}$. The configuration with minimum energy is therefore

$$\vec{\sigma}_{\vec{n}} = \frac{\vec{B}}{|\vec{B}|} \ , \quad \forall \vec{n}$$

The magnetization is an N-component vector given by the total vector sum of the expectation values of the individual spins,

$$\vec{M} \equiv \sum_{\vec{n}} < \vec{\sigma}_{\vec{n}} >$$

In the lowest energy state, it is equal to

$$\vec{M} = N^d \frac{\vec{B}}{|\vec{B}|}$$

where N^d is the total number of lattice sites. The magnetization density in this state is given by

$$\vec{m} = \frac{\vec{M}}{N^d} = \frac{\vec{B}}{|\vec{B}|}$$

and it is a constant. This is a Ferromagnetic state in that the magnetization does not go to zero, but it persists even as we take the limit $\vec{B} \to 0$. When $\vec{B} \to 0$, this becomes a state with spontaneously broken symmetry, in this case the $O(\mathcal{N})$ symmetry which rotates the $\mathcal{N}$-vectors $\vec{\sigma}$. This is analogous to the Z_2 symmetry breaking that we found in the Ising model. However, there is a big difference between this model and the Ising model, in that $O(\mathcal{N})$ is a continuous symmetry group. It has an infinitesimal transformation, which is just an infinitesimal rotation. This fact has some non-trivial consequences.

The ground state energy is given by

$$< H >= -JN^d d - |\vec{B}|N^d$$

This is a non-analytic function of the field $\vec{B}$ with the singular point being at $\vec{B} = 0$. The energy density is

$$u = \frac{< H >}{V} = \frac{-Jd}{a^d} - \frac{1}{a^d}|\vec{B}|$$

If we keep the temperature at $T = 0$ and vary the components of $\vec{B}$, there is a phase transition which occurs at the origin $\vec{B} = 0$ in the $\mathcal{N}$-dimensional space of possible values of $\vec{B}$.

The ground state here is similar to that of the Ising model in that all of the spins are aligned in a particular direction. However, the excitations of the two systems have significant differences. In the Ising model, the lowest energy excitation is found by just flipping one spin. The result of flipping one spin is $2d$ misaligned bonds which have energy $2dJ$. In the $O(\mathcal{N})$ model on the other hand, we could create an excitation which has arbitrarily low energy by misaligning one spin $\vec{\phi} \to \vec{\phi}'$ by an infinitesimal amount so that the excitation energy is $dJ\left[1 - \vec{\phi} \cdot \vec{\phi}'\right]$ which can be arbitrarily small. This turns out to make the symmetry broken state less stable in that it can be destroyed by large fluctuations of the low energy excitations. Indeed, at finite temperature, the order is destroyed by fluctuations in $d \leq 2$ and even at zero temperature, it is not stable at $d = 1$.

The partition function of this spin model is given by averaging the Boltzmann weight, $e^{-H/k_B T}$ over all of the possible spin configurations,

$$Z[T, N, \vec{B}] = \int \prod_{\vec{n}} d\vec{\sigma}_{\vec{n}}\ e^{\frac{J}{k_B T} \sum_{\vec{n},\hat{\mu}} \vec{\sigma}_{\vec{n}} \cdot \vec{\sigma}_{\vec{n}+\mu} + \frac{\vec{B}}{k_B T} \cdot \sum_{\vec{n}} \vec{\sigma}_{\vec{n}}}. \tag{14}$$

Here $\int d\sigma_{\vec{n}}$ is the volume integral on the $(\mathcal{N}-1)$-dimensional unit sphere which is the subspace of $\mathcal{N}$-dimensional Euclidean space, that is, the points $(\sigma^1, \sigma^2, \ldots, \sigma^N)$ which obey $(\sigma^1)^2 + (\sigma^2)^2 + \ldots + (\sigma^N)^2 = 1$. We will have more to say about how such an integral is done shortly.

As we did for the Ising model, we can use a Gaussian transformation to rewrite its partition function of the $O(\mathcal{N})$ spin model as that of a lattice field theory. To do that, consider the following identity

$$1 = \frac{\int [d\phi] e^{-\frac{1}{2}\sum_{\vec{x},\vec{y}} \Delta^{-1}(x,y)[\vec{\phi}(\vec{x}) - \sum_{\vec{w}} \Delta(x,w)\vec{\sigma}(\vec{w})]\cdot[\vec{\phi}(\vec{y}) - \sum_{\vec{z}} \Delta(y,z)\vec{\sigma}(\vec{z})]}}{\int [d\phi] e^{-\frac{1}{2}\sum_{\vec{x},\vec{y}} \Delta^{-1}(x,y)\vec{\phi}(\vec{x})\cdot\vec{\phi}(\vec{y})}}$$

$\Delta(x,y) = \Delta(n,n')$ (with $\vec{x} = a\vec{n}$ and $\vec{y} = a\vec{n}'$) is the same matrix that we used for the Ising model. Also

$$\int [d\phi]... \equiv \prod_{a=1}^{\mathcal{N}} \prod_{\vec{x}} \int_{-\infty}^{\infty} d\phi^a(\vec{x})...$$

Unlike the spin variables $\vec{\sigma}(\vec{x})$, which are constrained to be $\mathcal{N}$-dimensional unit vectors, the lattice field $\vec{\phi}(\vec{x})$ is simply an $\mathcal{N}$-dimensional vector each of whose components $\phi^a(\vec{x})$ can take on values which are any real numbers. The integration over each component $\phi^a(\vec{x})$ is over the entire real number line.

With the integral identity above, we can write the $O(\mathcal{N})$-model partition function as

$$Z[T,N,\vec{B}] = \frac{\int [d\phi] \exp\left(-S[\vec{\phi}] - \frac{N^d \mathcal{N} c}{2k_B T}\right)}{\left[(2\pi)^{N^d/2}\sqrt{\det \Delta}\right]^{\mathcal{N}}} \tag{15}$$

$$S[\vec{\phi}] = S_0[\vec{\phi}] + V[\vec{\phi}] \tag{16}$$

$$S_0[\vec{\phi}] = \frac{1}{2}\sum_{\vec{x},\vec{y}} \Delta^{-1}(x,y)\vec{\phi}(\vec{x})\cdot\vec{\phi}(\vec{y}) \tag{17}$$

$$V[\vec{\phi}] = \sum_{\vec{x}} V(\vec{\phi}(\vec{x})) \tag{18}$$

$$V(\vec{\phi}) = -\ln\left[\int d\vec{\sigma} \exp\left\{\left(\vec{\phi} + \frac{\vec{B}}{k_B T}\right)\cdot\vec{\sigma}\right\}\right] \tag{19}$$

$$\vec{m}(\vec{x}) \equiv <\vec{\sigma}(\vec{x})> = \sum_{\vec{y}} \Delta^{-1}(x,y) < \vec{\phi}(\vec{y}) > \tag{20}$$

$$\chi^{ab}(x,y) \equiv <\sigma^a(\vec{x})\sigma^b(y)>_C \tag{21}$$

$$= \sum_{\vec{w}\vec{z}} \Delta^{-1}(x,w) \left\langle\ \phi^a(\vec{w})\phi^b(\vec{z})\ \right\rangle_C \Delta^{-1}(z,y) - \Delta^{-1}(x,y)\delta^{ab} \tag{22}$$

The integral $\int d\vec{\sigma}...$ in equation (19) is over the $\mathcal{N}-1$-sphere.

Here, we have presented the potential $V(\vec{\phi})$ as an integral over a sphere. This is not an easy integral to get analytically. It is easy, on the other hand, to expand the potential in powers of $\vec{\phi}$ to a few orders (let us put $\vec{B}$ to zero for now),

$$V(\vec{\phi}) = -\ln\frac{\int d\vec{\sigma}\exp\left(\vec{\phi}\cdot\vec{\sigma}\right)}{\int d\vec{\sigma}} \quad -\ln\int d\vec{\sigma}$$

$$V(\vec{\phi}) = -\ln\left\{\frac{\int d\vec{\sigma}\left[1+\vec{\phi}\cdot\vec{\sigma}+\frac{1}{2!}(\vec{\phi}\cdot\vec{\sigma})^2+\dots\right]}{\int d\vec{\sigma}}\right\} \quad -\ln\int d\vec{\sigma}$$

Consider the following identities for integrals over the unit sphere:

$$\frac{\int d\vec{\sigma}\sigma^a}{\int d\vec{\sigma}} = 0\ ,\ \frac{\int d\vec{\sigma}\sigma^a\sigma^b}{\int d\vec{\sigma}} = \frac{1}{\mathcal{N}}\delta^{ab}\ ,\ \frac{\int d\vec{\sigma}\sigma^a\sigma^b\sigma^c}{\int d\vec{\sigma}} = 0\ ,$$

$$\frac{\int d\vec{\sigma}\sigma^a\sigma^b\sigma^c\sigma^d}{\int d\vec{\sigma}} = \frac{1}{\mathcal{N}(\mathcal{N}+2)}\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right)$$

$$\frac{\int d\vec{\sigma}\sigma^a\sigma^b\sigma^c\sigma^d\sigma^e\sigma^f}{\int d\vec{\sigma}} = \frac{1}{\mathcal{N}(\mathcal{N}+2)(\mathcal{N}+4)}\left(\delta^{ab}\delta^{cd}\delta^{ef}+\delta^{ac}\delta^{bd}\delta^{ef}+\delta^{ad}\delta^{bc}\delta^{ef}+\dots\right)$$

The general identity of this kind is found by considering the Gaussian integral over Euclidean space,

$$\int d\vec{p}e^{-\vec{p}^2/2}p^{i_1}p^{i_2}\dots p^{i_{2k}} = \lim_{j\to 0}\frac{\partial}{\partial j_{i_1}}\cdots\frac{\partial}{\partial j_{i_{2k}}}\int d\vec{p}e^{-\vec{p}^2/2+\vec{p}\cdot\vec{j}} = \lim_{j\to 0}\frac{\partial}{\partial j_{i_1}}\cdots\frac{\partial}{\partial j_{i_{2k}}}(2\pi)^{\mathcal{N}/2}e^{\vec{j}^2/2}$$

$$= (2\pi)^{\mathcal{N}/2}\sum_{\text{pairings}}\prod_{\text{pairs(a,b)}}\delta^{i_a i_b}$$

where the sum is over distinct pairings of the $2k$ indices. Then we can also do the integral by going to polar coordinates, $\vec{p}=p\vec{\sigma}$ with $\vec{\sigma}$ a unit vector, so that

$$\int d^{\mathcal{N}}pe^{-\vec{p}^2/2}p^{i_1}p^{i_2}\dots p^{i_{2k}} = \int_0^\infty p^{\mathcal{N}-1+2k}dpe^{-p^2/2}\int d\vec{\sigma}\sigma^{i_1}\sigma^{i_2}\dots\sigma^{i_{2k}}$$

$$= 2^{\mathcal{N}/2-1+k}\int_0^\infty (p^2/2)^{\mathcal{N}/2-1+k}d(p^2/2)e^{-p^2/2}\int d\vec{\sigma}\sigma^{i_1}\sigma^{i_2}\dots\sigma^{i_{2k}}$$

$$= 2^{\mathcal{N}/2-1+k}\Gamma[\mathcal{N}/2+k]\int d\vec{\sigma}\sigma^{i_1}\sigma^{i_2}\dots\sigma^{i_{2k}}$$

Equating the two equations gives us

$$\int d\hat{\sigma}\hat{\sigma}^{i_1}\hat{\sigma}^{i_2}\dots\hat{\sigma}^{i_{2k}} = \frac{(2\pi)^{\mathcal{N}/2}}{2^{\mathcal{N}/2-1+k}\Gamma[\mathcal{N}/2+k]}\sum_{\text{pairings}}\prod_{\text{pairs(a,b)}}\delta^{i_a i_b}$$

Note that this also tells us the volume of the $\mathcal{N}-1$-sphere,

$$\text{volume }(S^{\mathcal{N}-1}) \;=\; \int d\hat{\sigma} = \frac{(2\pi)^{\mathcal{N}/2}}{2^{\mathcal{N}/2-1}\Gamma[\mathcal{N}/2]}$$

2. *Confirm that the above expression for the volume of the $\mathcal{N}-1$ sphere, $S^{\mathcal{N}-1}$, for the volume of S^1, S^2, S^3 which are $2\pi, 4\pi, 2\pi^2$, respectively.*

   We can find an expression for the ratio of integrals

$$\frac{\int d\hat{\sigma}\hat{\sigma}^{i_1}\vec{\sigma}^{i_2}\dots\hat{\sigma}^{i_{2k}}}{\int d\vec{\sigma}} = \frac{\Gamma[\mathcal{N}/2]}{2^k\Gamma[\mathcal{N}/2+k]}\sum_{\text{pairings}}\prod_{\text{pairs(a,b)}}\delta^{i_a i_b}$$

   or, using the property of the Euler gamma function $\Gamma[x]=(x-1)\Gamma[x-1]$,

$$\frac{\int d\hat{\sigma}\hat{\sigma}^{i_1}\vec{\sigma}^{i_2}\dots\hat{\sigma}^{i_{2k}}}{\int d\vec{\sigma}} = \frac{1}{\mathcal{N}(\mathcal{N}+2)\dots(\mathcal{N}+2k-2)}\sum_{\text{pairings}}\prod_{\text{pairs(a,b)}}\delta^{i_a i_b}$$

   Finally, by counting the pairings we see that

$$\frac{\int d\vec{\sigma}(\vec{\phi}\cdot\vec{\sigma})^{2k}}{\int d\vec{\sigma}} = \frac{1}{\mathcal{N}(\mathcal{N}+2)\dots(\mathcal{N}+2k-2)}\frac{2k!}{2^k k!}|\vec{\phi}|^{2k}$$

   and

$$\frac{\int d\vec{\sigma}\exp(\vec{\phi}\cdot\vec{\sigma})}{\int d\vec{\sigma}} = \sum_{k=0}^{\infty}\frac{1}{\mathcal{N}(\mathcal{N}+2)\dots(\mathcal{N}+2k-2)}\frac{1}{2^k}|\vec{\phi}|^{2kk!}$$

3. *Show that the leading terms in the Taylor expansion of the potential are*

$$V(\vec{\phi}) = -\frac{1}{2\mathcal{N}}\vec{\phi}^2 + \frac{1}{4\mathcal{N}^2[\mathcal{N}+2]}(\vec{\phi}^2)^2 \;+\; \dots \;-\; \ln\int d\vec{\sigma}$$

Can you get the order $(\vec{\phi}^2)^3$ term?

Our goal is to study the partition function and the Helmholtz free energy of the $O(\mathcal{N})$ model. The first approximation that we will take is motivated that the idea that the interesting behaviour of this integral comes from the very long wave-length modes of the lattice function $\phi(\vec{x})$, that is, the modes of the function are almost independent of $\vec{x}$. Then, as a first attempt, we simply replace $\vec{\phi}(\vec{x})$ by a constant $\mathcal{N}$-vector $\vec{\phi}$ and integrate over it.

4. *Show that, when we assume that the lattice field $\vec{\phi}(\vec{x})$ has only constant components, and we put the external field $\vec{B} \to 0$, the partition function is given by the expression*

$$Z[T,N] = \left[\frac{1}{(2\pi)^{N^d/2}\sqrt{\det\Delta}}\right]^{\mathcal{N}} \int_{-\infty}^{\infty} d\vec{\phi}\exp\left(-S_1[\vec{\phi}] - \frac{cN^d\mathcal{N}}{2k_BT}\right) \quad (23)$$

where

$$S_1[\vec{\phi}] = \frac{\vec{\phi}^2}{2}\sum_{\vec{x},\vec{y}}\Delta^{-1}(x,y) + N^dV(\vec{\phi})$$

5. *Show that the quantity*

$$\sum_{\vec{x}\vec{y}}\Delta^{-1}(x,y) = N^d\,\frac{k_BT}{2dJ+c} = N^d\frac{T}{T_c\mathcal{N}}$$

where we have defined $T_c \equiv (2dJ+c)/\mathcal{N}k_B$.

We then find that

$$S_1[\vec{\phi}] = N^d\left\{\frac{T}{T_c}\,\frac{\vec{\phi}^2}{2\mathcal{N}} + V(\vec{\phi})\right\}$$

When N^d is large, we can use the saddle-point technique to do the integral over $\vec{\phi}$ to get the partition function and the Helmholtz free energy

$$F[T,N] = N^dk_BT\inf_{\vec{\phi}}\left\{\frac{T}{T_c}\frac{\vec{\phi}^2}{2\mathcal{N}} + V(\vec{\phi})\right\} - N^d\mathcal{N}\tilde{c}$$

where the constant $\tilde{c}$ takes into account both the constant c and the quantities coming from the factor $1/N^{d/2}\sqrt{\det\Delta}$ in the partition function. $\tilde{c}$ turns out to be independent of both T and N. The symbol inf means that we should find the infimum of the function of $\vec{\phi}$ that sits to the right of it.

We should recognize the expression for the Free energy to be that of mean field theory. The approximation where we retain only the constant part of the field $\vec{\phi}(\vec{x})$ is equivalent to mean field theory. In this approximation, all of the critical exponents will be those of mean field theory. In the following, we will study a few of the details.

6. *Find the value of $\vec{\phi}$ that realizes the infimum in equation (24) and show that, when it is plugged into the Helmholtz free energy, the Free energy density, defined as*

$$f(T) \equiv \lim_{N\to\infty} \frac{1}{N^d a^d} F[T,N]$$

is given by the expression

$$f(T) = \begin{cases} \frac{k_B T}{a^d} V(\vec{0}) - c\mathcal{N}/a^d & T > T_c \\ -\frac{k_B T}{a^d}(\mathcal{N}+2)(1-T/T_c)^2 + \ldots + \frac{k_B T}{a^d} V(\vec{0}) - c\mathcal{N}/a^d & T < T_c \end{cases} \tag{24}$$

where the ellipses denote contributions from higher integer powers of $(1-T/T_c)$ – so that the terms that we have included become arbitrarily accurate as we take $T \to T_c$.

7. *What is the order of the phase transition at $T = T_C$?*

8. *Show that the magnetization is given by*

$$\vec{m} = \frac{T}{T_c}\sqrt{\frac{\mathcal{N}+2}{\mathcal{N}}}\sqrt{1-T/T_c}\ \hat{e}$$

where $\hat{e}$ is an $\mathcal{N}$-component unit vector. The orientation of the magnetization is in any direction in the $\mathcal{N}$-dimensional space. This corresponds to spontaneous symmetry breaking. Unbroken $O(\mathcal{N})$ symmetry would average the magnetization to zero. Here, it is nonzero. Some of the symmetry remains. That symmetry contains all of the rotations in

$\mathcal{N}$-dimensional space which would leave the magnetization unchanged. This is an $O(\mathcal{N}-1)$ symmetry. We say that the $O(\mathcal{N})$ symmetry is broken to $O(\mathcal{N}-1)$.

In order to examine the validity of mean field theory, it is instructive to attempt to study the leading corrections to it. With that aim, we will begin by inserting the following identity into the integral which defines the partition function,

$$1 = \int d\vec{\varphi}\ \delta\left(\vec{\varphi} - \frac{1}{N^d}\sum_{\vec{x}} \vec{\phi}(\vec{x})\right)$$

Here, $\int d\vec{\varphi}$ is an integration over $\mathcal{N}$-dimensional space.

9. *Show that the above identity can be used to put the partition function in the form*

$$Z[T,N] = \int d\vec{\varphi} \left[\frac{1}{(2\pi)^{N^d/2}\sqrt{\det\Delta}}\right]^{\mathcal{N}} \int [d\phi]\delta\left(\frac{1}{N^d}\sum_{\vec{x}} \vec{\phi}(\vec{x})\right)\exp\left(-S[\vec{\varphi}+\vec{\phi}]\right) \tag{25}$$

Now, we can expand the exponential in equaiton (25) up to second order and beyond in the integration variable $\vec{\phi}(\vec{x})$:

$$\begin{aligned} S[\vec{\varphi}+\vec{\phi}] = & \frac{T}{T_c}\frac{\vec{\varphi}^2}{2\mathcal{N}} + V(\vec{\varphi}) + \frac{\partial V(\vec{\varphi})}{\partial \varphi^a}\sum_{\vec{x}} \phi^a(\vec{x}) \\ & + \frac{1}{2}\sum_{\vec{x},\vec{y}} \phi^a(\vec{x})\left(\Delta^{-1}(x,y)\delta^{ab} + \frac{\partial^2 V(\vec{\varphi})}{\partial\varphi^a\partial\varphi^b}\right)\phi^b(\vec{y}) + \ldots \end{aligned}$$

10. *Use this expansion up to quadratic order and do the Gaussian integral to rewrite the partition function as*

$$Z[T,N] = \int d\vec{\varphi}\left[\frac{1}{\sqrt{\det\Delta}}\right]^{\mathcal{N}} \frac{\exp\left(-\frac{V}{a^d}\left[\frac{T}{T_c}\frac{\vec{\varphi}^2}{2\mathcal{N}} + V(\vec{\varphi}) - \frac{\mathcal{N}c}{2k_B T}\right]\right)}{\sqrt{\det\left(\Delta^{-1}(x,y)\delta^{ab} + \frac{\partial^2 V(\vec{\varphi})}{\partial\varphi^a\partial\varphi^b}\right)}} \tag{26}$$

11. *Use the fact that*

$$\frac{\partial^2 V(\vec{\varphi})}{\partial\varphi^a\partial\varphi^b} = \frac{\partial}{\partial\varphi^b}\frac{\varphi^a}{|\varphi|}V'(|\vec{\varphi}|) = \delta^{ab}\frac{V'(|\vec{\phi}|)}{|\vec{\phi}|} + \frac{\varphi^a\varphi^b}{\vec{\varphi}^2}\left(V''(|\varphi|) - \frac{V'(\varphi|)}{|\varphi|}\right)$$

to show that $\Delta^{-1}(x,y)\delta^{ab} + \frac{\partial^2 V(\vec{\varphi})}{\partial\varphi^a\partial\varphi^b}$ *has, for each wave-vector* $\vec{k}$, $\mathcal{N}-1$ *eigenvalues*

$$\Delta^{-1}(\vec{k}) + \frac{V'(|\vec{\phi}|)}{|\vec{\phi}|}$$

and one eigenvalue

$$\Delta^{-1}(\vec{k}) + V''(|\varphi|)$$

12. *Show that the expression that you obtained for the partition function in equation (26) leads to the following expression for the Helmholtz free energy density*

$$f(T) = \tag{27}$$

$$k_B T \inf_{\varphi}\left\{\frac{1}{a^d}\left[\frac{T}{T_c}\frac{\vec{\varphi}^2}{2\mathcal{N}} + V(\vec{\varphi})\right] + \frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^d}\ln\frac{\left(\Delta^{-1}(\vec{k}) + \frac{V'(|\vec{\phi}|)}{|\vec{\phi}|}\right)^{\mathcal{N}-1}\left(\Delta^{-1}(\vec{k}) + V''(|\varphi|)\right)}{\left(\Delta^{-1}(\vec{k})\right)^{\mathcal{N}}}\right\} \tag{28}$$

Now, we will do our second truncation of the degrees of freedom. We will assume that the only important modes are those with wave-numbers $\vec{k}$ in the regime $0 \le |\vec{k}| \le \Lambda$ where Λ is called the "'cutoff" and $\Lambda << \frac{1}{a}$.

The integral that remains to be done is

$$\int_{|\vec{k}|\le\Lambda}\frac{d\vec{k}}{(2\pi)^d}\ln\frac{\left(\Delta^{-1}(\vec{k}) + \frac{V'(|\vec{\phi}|)}{|\vec{\phi}|}\right)^{\mathcal{N}-1}\left(\Delta^{-1}(\vec{k}) + V''(|\varphi|)\right)}{\left(\Delta^{-1}(\vec{k})\right)^{\mathcal{N}}}$$

where we have imposed the cutoff.

13. *Show that the cutoff integral can be well approximated by the expression*

$$\int_{|\vec{k}|\le\Lambda}\frac{d\vec{k}}{(2\pi)^d}\ln\frac{\left(\Delta^{-1}(\vec{k}) + \frac{V'(|\vec{\phi}|)}{|\vec{\phi}|}\right)^{\mathcal{N}-1}\left(\Delta^{-1}(\vec{k}) + V''(|\varphi|)\right)}{\left(\Delta^{-1}(\vec{k})\right)^{\mathcal{N}}}$$

$$= \int_{|\vec{k}|\le\Lambda} \frac{d\vec{k}}{(2\pi)^d} \ln \frac{\left(\vec{k}^2 + \mu^2 + \frac{T_c}{T}\mu^2\mathcal{N}\frac{V'(|\vec{\phi}|)}{|\vec{\phi}|}\right)^{\mathcal{N}-1}\left(\vec{k}^2 + \mu^2 + \frac{T_c}{T}\mu^2\mathcal{N}V''(|\varphi|)\right)}{\left(k^2+\mu^2\right)^{\mathcal{N}}}$$

where

$$\mu^2 = \frac{2Jd + c}{Ja^2}$$

when $\Lambda a << 1$.

When d is not equal to an even non-negative integer, the integral in the equation above is well-defined and convergent even without the cutoff Λ. We can use this fact to add back in some irrelevant modes, those with wave-vectors with magnitudes larger that Λ to write another integral which should have the same non-analytic behaviour,

$$\int \frac{d\vec{k}}{(2\pi)^d} \ln \frac{\left(\vec{k}^2 + \mu^2 + \frac{T_c}{T}\mu^2\mathcal{N}\frac{V'(|\vec{\varphi}|)}{|\vec{\varphi}|}\right)^{\mathcal{N}-1}\left(\vec{k}^2 + \mu^2 + \frac{T_c}{T}\mu^2\mathcal{N}V''(|\vec{\varphi}|)\right)}{\left(k^2+\mu^2\right)^{\mathcal{N}}}$$

14. *Evaluate the integral in the expression above and use your result to show that the free energy density is given by*

$$f(T) = \frac{k_B T}{a^d} \inf_{\varphi} \left\{ \frac{T}{T_c}\frac{|\vec{\varphi}|^2}{2\mathcal{N}} + V(|\vec{\varphi}|) - \frac{c}{2k_B T} \right.$$
$$\left. - \frac{\Gamma[-d/2](a\mu)^d}{2(4\pi)^{d/2}} \left[(\mathcal{N}-1)\left(1 + \frac{T_c}{T}\mathcal{N}\frac{V'(|\vec{\varphi}|)}{|\vec{\varphi}|}\right)^{d/2} + \left(1 + \frac{T_c}{T}\mathcal{N}V''(|\varphi|)\right)^{d/2} - \mathcal{N} \right] \right\} \tag{29}$$

15. *Find the equations for the value of $\vec{\varphi}$ at which the infimun in equation (29) is achieved. (You will need to use the Taylor expansion of the potential in $\vec{\phi}^2$ to a sufficiently high order.)*

16. *Find a solution of the equations for φ in the above equation for a small range of temperatures both above and below the critical temperature. When there are multiple solutions, identify which solution gives the infimum.*

17. *Plug solutions that you found in the previous problem back into equation (29) to find the free energy density as a function of temperature. Note that the result when $d > 4$ has significant differences from that where $d \le 4$.*