# Critical Exponents and the Renormalization Group

Marina Maciel Ansanelli

December 2019

## Contents



## 1 Introduction

In these notes, we are going to study problems in Statistical Mechanics that involve *phase transitions*, the abrupt changes that occur to a system when we reach specific values of certain parameters. Our main point here is to apply to these problems the Renormalization Group formalism that we learned in QFT. As we'll see, the Renormalization Group gives us a powerful tool to study the behavior of a lot of different systems when they're close to the phase transition, and shows us why the first approximation we can think of (the Mean Field Approximation) fails at low dimension.

First of all, we should briefly see *why* we can trace a parallel between Quantum Field Theory and Statistical Mechanics. In Quantum Field Theory, the generating functional of the correlation functions is written as the path integral:

$$Z[J] = Z[0] \int \mathcal{D}\varphi \, \exp\left\{ \frac{i}{\hbar} \left( S[\varphi] + \int d^4x J(x)\varphi(x) \right) \right\}$$

where $S$ is the action and $J$ is an external source.

If we perform a rotation on the complex plane going to imaginary times $x_4 = ix_0$, we end up with the *Euclidean* generating functional:

$$Z[J] = Z[0] \int \mathcal{D}\varphi \, \exp\left\{ \frac{-1}{\hbar} \left( S_E[\varphi] + \int d^4x_E J(x_E)\varphi(x_E) \right) \right\} \tag{1}$$

with a real euclidean action $S_E \geq 0$. This result has its convergence guaranteed.

In classical Statistical Mechanics, the probability distribution of a thermal state is written as

$$\rho_{\text{thermo}}(\phi) = \frac{e^{-\beta E(\phi)}}{Z} \ , \ Z = \sum_{\text{all states}} e^{-\beta E(\phi)} \tag{2}$$

There is a clear analogy between equations (1) and (2), that will be made more explicit when we coarse grain the statistical mechanics random variable to write it as a continuous function in section 2.3.

# 2 Critical Phenomena

We'll focus on a specific example that's very instructive: the ferromagnetic Ising Model. Although it looks too specific, it turns out that the behavior near the phase transition is the same for a lot of different systems, a phenomenon that looks kind of mysterious by now but will be justified later, known as *universality*. In particular, it's the same for the Ising ferromagnet and for a Van-der-Waals gas. So, we first introduce this simple model. For this section, our main references are [1] and [2].

## 2.1 The Ising Model

The model consists of a lattice in d spatial dimensions, with N sites. Each site has a variable that can take values $s_i = \pm 1$. We are going to refer to this variable as the *spin*, but the model we consider in these notes is essentially classical; the $s_i$'s are not operators, but just numbers that take discrete values.

The Hamiltonian of the configuration is

$$H(\{s_i\}) = -h \sum_{i=1}^{N} s_i - J \sum_{\langle ij \rangle} s_i s_j \tag{3}$$

where $\langle ij \rangle$ means that we're summing over nearest neighbors, and $h$ is an external magnetic field (the analog to the source $J(x_E)$ in equation (1)). Here we'll assume $J > 0$, so the spins tend to align: we have a ferromagnet. As we will see, the Ising model have two types of phase transitions: *second order* phase transitions, that happen when we reach a critical temperature, and *first order* phase transitions, that happen at zero temperature, when we vary the external field $h$.

In equilibrium statistical mechanics, our task is to evaluate the partition function

$$Z(\beta, h, N) = \sum_{\{s_i\}} e^{-\beta H\{s_i\}} \tag{4}$$

If we could separate the Hamiltonian of equation (3) in terms that depend only on the spin $s_i$, then $Z$ would be just the partition function of one spin (that is easy to compute) to the power of $N$. However, the interaction term, which coupling is $J$, spoils this. So, our first try is to make an approximation such that the spins are independent.

## 2.2 Mean Field Approximation

In this specific subsection, we follow [3].

The Mean Field Approximation consists of considering that the fluctuations of the spins around the average are small. The *equilibrium* magnetization is given by $m_{eq} = \frac{1}{N} \langle \sum_i s_i \rangle = \langle s_i \rangle$, where $\langle . \rangle$ means the average over all the spin configurations in the canonical ensemble. It is independent of $i$ because we assume translational invariance. The fluctuation is

$$\delta s_i = s_i - \langle s_i \rangle = s_i - m$$

where we used $m$ to avoid the heavy notation, but we are actually dealing with the equilibrium magnetization $m_{eq}$.

We want to decouple the spins in the $J$ term of equation (3). So, we write $s_i s_j$ as:

$$s_i s_j = (\delta s_i + m)(\delta s_j + m) = m^2 + m(\delta s_i + \delta s_j) + \delta s_i \delta s_j \tag{5}$$

Here is where the approximation enters: we say that the term $\delta s_i \delta s_j$ is small an can be neglected. Keeping the other terms, equation (5) becomes:

$$s_i s_j = m^2 + m(s_i - m + s_j - m) = m(s_i + s_j) - m^2$$

$$\sum_{\langle ij \rangle} s_i s_j = m\nu \sum_i s_i - \frac{N\nu}{2} m^2$$

where $\nu = 2d$ is the number of first neighbors in our lattice, so $N\nu/2$ is the number of bonds. Using it, the Hamiltonian in equation (3) becomes:

$$H(\{s_i\}) = -(J\nu m + h) \sum_{i=1}^{N} s_i + J\frac{N\nu}{2} m^2 \tag{6}$$

With that, we can get the partition function of equation (4) by separating the Hamiltonian:

$$Z = \left( \sum_{s=\pm 1} e^{\frac{-\beta J \nu m^2}{2}} e^{\beta(J\nu m + h)s} \right)^N = e^{\frac{-\beta N J \nu m^2}{2}} \left(2 \cosh \beta(J\nu m + h)\right)^N \tag{7}$$

We can find the equilibrium magnetization using the relations

$$m_{eq}(T, h) = -\frac{1}{N} \left( \frac{\partial F_{eq}}{\partial h} \right)_{T,N} , \tag{8}$$

$$F_{eq}(T, h, N) = -\frac{1}{\beta} \ln Z \tag{9}$$

and the result is:

$$\boxed{m_{eq}(T, h) = \tanh\left(\beta h + \beta J \nu m_{eq}\right)} \tag{10}$$

That's a self-consistent equation whose solutions show phase transition behavior.

### 2.2.1 Critical Behavior in the MF-Approximation

Setting $h = 0$ in equation (10), the existence of any non-zero solution for $m$ requires that $\beta J\nu > 1$ (figure 1). So, we can already identify the *critical temperature* within mean field approximation: $T_C = J\nu$. At this temperature, our system passes through a *second-order* phase transition: the magnetization $m$ changes continuously (figure 2), but is derivative changes discontinuously, in the $N \to \infty$ limit. We call the region $T > T_C$ the *disordered* phase, and the region $T < T_C$ the *ordered* phase.

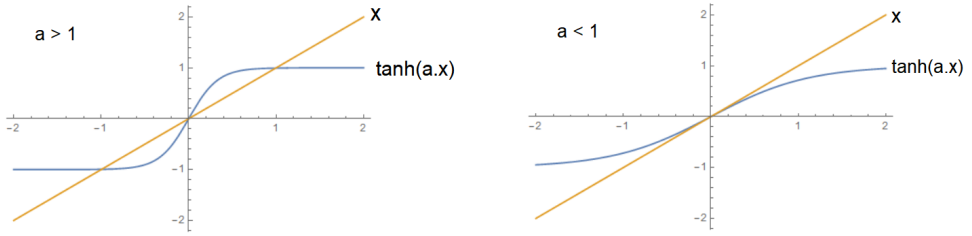



Figure 1: Examples of the comparison between the functions $x$ and $\tanh ax$ for values of $a$ smaller and greater than zero. Figure made with Mathematica.

If we are in the vicinity of the fixed point, we can use the expansion $\tanh^{-1} x \approx x + x^3/3$ to see that the approximate solution of (10) is either $m_{eq} = 0$ or

$$m_{eq} = \pm\sqrt{\frac{3(T_C - T)}{T}} \sim |T - T_C|^{1/2} \tag{11}$$

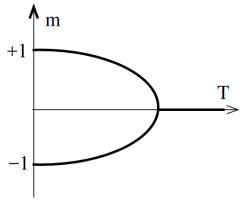



Figure 2: Second order phase transition at temperature $T = T_C$, with $h = 0$. Figure taken from [1].

This 1/2 is the first mean-field *critical exponent* that we encounter. Still at $h = 0$, we can calculate the heat capacity:

$$C = \beta^2 \frac{\partial^2}{\partial \beta^2} \log Z \tag{12}$$

Performing the calculation with the partition function given by equation (7) and using the magnetization given by (11) for $T < T_C$, the result in the limit $T \to T_C$ is:

$$c = \frac{C}{N} \xrightarrow{T \to T_C} \begin{cases} 0 & T > T_C \\ 3/2 & T < T_C \end{cases} \tag{13}$$

The heat capacity indeed changes discontinuously.

Now, we turn to the first-order phase transition that happens in the Ising model when we're at a fixed temperature $T < T_C$ and we change the external magnetic field. The origin of this phase transition can be seen in figure 3.

The external magnetic field destroys the second order phase transition at $T = T_C$, because the sign of the magnetization is determined by the sign of $h$. So, when we vary $h$ at constant $T < T_C$, the magnetization $m$ discontinuously jumps to $-m$ when $h$ changes sign: we have a first-order phase transition.

Associated to this phase transition we can also find other two critical exponents. The first one shows how $m_{eq}$ varies with $h$ near $h = 0$:

$$h = \frac{1}{\beta} \tanh^{-1} m - mT_C \approx (T - T_C)m + Tm^3/3$$

We could do this expansion because we are assuming that $m$ is small: for this to be true, we must be close to $h = 0$ and to $T = T_C$. If $T$ is close to $T_C$, we ignore the first term in the above expression and see that the magnetization scales with the external field as

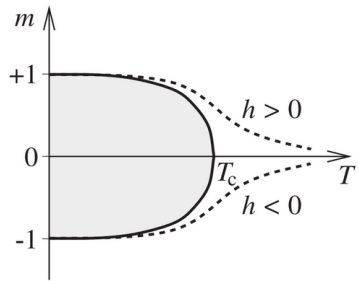



Figure 3: Magnetization at different values of the external field $h$. Figure taken from [2].

$$m \sim h^{1/3} \tag{14}$$

We can also ask how the magnetic susceptibility changes, $\chi = \frac{\partial m}{\partial h}|_T$. As it turns out, the scaling law is

$$\chi \sim \frac{1}{|T - T_C|} \tag{15}$$

So, when collecting equations (11), (13), (14) and (15), we have four mean-field critical exponents that define scaling laws around the critical point. We'll give the following names to these critical exponents:

$$\begin{aligned} m &\sim |T - T_C|^{\beta} & \beta &= \frac{1}{2} \\ c &\sim c_{\pm} |T - T_C|^{-\alpha} & \alpha &= 0 \\ m &\sim h^{1/\delta} & \delta &= 3 \\ \chi &\sim \frac{1}{|T - T_C|^{\gamma}} & \gamma &= 1 \end{aligned}$$

All these quantities are measurable, and can be compared to experiment. More than that, the Ising model can be analytically solved for $d \leq 2$ and numerically approximated to higher dimensions. As it turns out, **for dimensions greater or equal than 4, the mean field results are exact**. However, for $d = 2$ and $d = 3$ it gives the wrong critical exponents, and for $d = 1$ it fails completely: there is no phase transitions.

Up to now, this breakdown of mean field theory at low dimensions seems a little mysterious. It's qualitatively reasonable that this approximation works best for higher dimensions, because the spin

interacts with more neighbors and so experiences something closer to the average. However, it's not yet explained what is so special about dimension four. We'll come back to it in section 4, with an explanation using arguments from the Renormalization group.

Before that, we must write down a simple generalization of what we've been doing, to see more clearly the analogy with Quantum Field Theory and to introduce the first steps that relate Statistical Field Theory to the Renormalization Group procedure.

## 2.3 Coarse-Graining

The Mean Field Approximation that we studied in the last section ruled out the possibility of strong spatial fluctuations in the spin lattice. Here, we generalize the idea to include the possibility of spatial variations by promoting the order parameter $m$ to a field $m(\boldsymbol{x})$ that can vary in space, and study how the Hamiltonian is constructed for such a model.

In this section, we will work with the *average magnetization of a given spin configuration*, given by $m = \frac{1}{N}\sum_i s_i$. Note that this is *not* the same as the equilibrium magnetization we worked with in section 2.2, although we used the letter $m$ back then sometimes. The equilibrium magnetization is the average of the magnetization over all configurations in the canonical ensemble (thermal equilibrium). This is the one that can be found by relation (8) (notice that it does not depend on a specific spin configuration of the system). To promote the magnetization to a field, we divide our lattice in many boxes with side length $a$. Each box contains many sites (say $N'$), but it is still much smaller than the scales we are going to observe in the experiment. Then, we calculate the average magnetization for each of these boxes, $m(\boldsymbol{x}) = \frac{1}{N'}\sum_i s_i$, and assign its value to the coordinate of the center of the box. This process is known as *coarse-graining*. Here, we want to treat $m(\boldsymbol{x})$ as a continuous function: we will consider that the number of boxes $N/N'$ is really big, and the number of sites in each box $N'$ is also big (so $m(\boldsymbol{x})$ can take any value from $-1$ to 1).

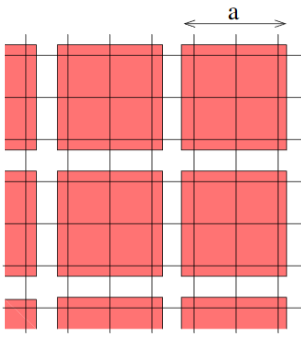



Figure 4: Coarse-graining of the lattice in box of size $a$. Figure taken from [1].

Now, we refer back to the partition function in equation (4). It involved a sum over all possible spin configurations for the lattice, that can be rewritten as a sum over all $m(\boldsymbol{x})$ and a sum over all spin configurations $\{s_i\}|m(\boldsymbol{x})$, that give rise to the same value of magnetization $m(\boldsymbol{x})$.

$$Z = \sum_{\{s_i\}} e^{-\beta E[s_i]} = \sum_{m(\boldsymbol{x})} \sum_{\{s_i\}|m(\boldsymbol{x})} e^{-\beta E[s_i]}$$

from here, we can define the *effective free energy* related to the magnetization field $m(\boldsymbol{x})$, a concept that will be very important in relating the phase transitions problem with the framework of the Renormalization Group.

$$e^{-\beta F[m(\boldsymbol{x})]} \equiv \sum_{\{s_i\}|m(\boldsymbol{x})} e^{-\beta E[s_i]} \quad \Rightarrow \quad Z = \sum_{m(\boldsymbol{x})} e^{-\beta F[m(\boldsymbol{x})]} \tag{16}$$

this is *not* the same as the equilibrium free energy given by equation (9): the equilibrium free energy is the effective free energy evaluated at the equilibrium magnetization or, in other words, the minimum of $F[m(\boldsymbol{x})]$.

In this section, the main point is to assume that the magnetization is actually a continuous function. So, the sum over all field configurations in equation (16) needs to be replaced by a path integral:

$$Z = \int \mathcal{D}m(\boldsymbol{x}) e^{-\beta F[m(\boldsymbol{x})]} \tag{17}$$

The assumption of locality (that comes from nearest neighbor interactions) in the Ising Model shows us that the free energy can be written as:

$$F[m(\boldsymbol{x})] = \int d^d x f[m(\boldsymbol{x})]$$

with $f[m(\boldsymbol{x})]$ being a local function.

With this at hand, we see clearly the analogy between Statistical Field Theory and Quantum Mechanics that was introduced in equations (1) and (2). Both of the theories can be described by partition functions that are written as path integrals:

$$\text{Statistical Field Theory: } Z = \int \mathcal{D}m e^{-\beta \int d^d x f[m]} \tag{18}$$

$$\text{Euclidean Quantum Field Theory: } Z = \int \mathcal{D}\phi e^{\frac{-1}{\hbar} \int d^d x \mathcal{L}(\phi)} \tag{19}$$

and the parameter that regulates the thermal fluctuations in statistical mechanics, the temperature, appears as analogous to the "parameter" that regulates the relevance of quantum fluctuations in quantum field theory, $\hbar$.

Now, we will give a brief review on the Renormalization Group formalism, to finally relate it to all we've been doing in this section.

# 3 The Renormalization Group - Brief Review

Throughout this section, our main reference is [4].

## 3.1 Wilsonian Renormalization Group

When we study critical phenomena, we are interested in the long distance behavior of our system, and usually ignore the short-range specific features. This is what we did in the coarse-graining procedure introduced in section 2.3, where we literally averaged out the small scales.

This is the basic idea for the Wilsonian Renormalization Group: we integrate out the large momenta (small distance) and pay attention only at the low momenta behavior of our system. In statistical mechanics spin lattices, as we studied, this procedure has a physical and clear reason: the sites in the lattice are discrete, so we have a real cut-off $\Lambda$ in the momentum. In quantum field theory, the cut-off is one of the possibilities for renormalizing the theory and getting rid of the infinities; however, unlike in the spin lattices, our final goal is to take $\Lambda \to \infty$. More than that, the cut-off method is not usually appropriate for quantum field theories when we want to study the Ward identities. However, we will develop the discussion of this section (and the next) in the quantum field theory framework.

In these notes, we are going to work only with scalar theories. If we start with a $\phi^4$ theory with an initial cut-off $\Lambda$, our partition function is

$$Z = \int [\mathcal{D}\phi]_\Lambda e^{\frac{-1}{\hbar} \int d^d x \mathcal{L}(\phi)} \tag{20}$$

where $[\mathcal{D}\phi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k)$, and we are in euclidean space.

Now, we choose a number $b < 1$ and redefine $\phi$ as being the low momentum field: it is equal to the original $\phi$ when $|k| < b\Lambda$, but it is zero at higher momentum. Equivalently, we define $\hat{\phi}$ that is zero at low momentum and equals the original one in $b\Lambda \leq |k| < \Lambda$. The original $\phi$ just equals the new $\phi$ plus $\hat{\phi}$, and we can rewrite $Z$ in terms of these new fields. Doing so, we will find in the exponent a lot of terms that mix $\phi$ and $\hat{\phi}$, and also some terms that depend only on $\phi$ or $\hat{\phi}$.

Now, the point is to integrate out the high momentum field $\hat{\phi}$ and end up with a path integral on the low momentum field:

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} e^{\frac{-1}{\hbar} \int d^d x \mathcal{L}_{eff}(\phi)} \tag{21}$$

We are not going to do the full development here, but it turns out that $\mathcal{L}_{eff}$ can be found writing down Feynman Rules to the field $\hat{\phi}$, considering the terms of the original Lagrangian that depend on $\hat{\phi}$ as perturbations. This development is made in [4]. The effective Lagrangian end up having all possible orders of operators in $\phi$, even those including derivatives.

To compare the expressions 20 and 21, we need them to be integrated up to the same momentum. So, we do the rescaling:

$$k' = k/b \qquad\qquad x' = xb$$

If the coefficient of $(\partial_\mu\phi)^2$ in $\mathcal{L}_{eff}$ is $(1+\Delta Z)/2$, we will also rescale $\phi$ for this coefficient to be $1/2$ after the full rescaling:

$$\phi' = [b^{2-d}(1+\Delta Z)]^{1/2}\phi$$

In the end, the procedure of integrating out high momenta and rescaling ends up being just a transformation $\mathcal{L} \to \mathcal{L}_{eff}$ in the Lagrangian. If $b$ is close to 1, this transformation becomes continuous: a flow in the space of all possible Lagrangians (an infinite-dimensional space whose axes are the coefficients of each possible term).

We have defined our transformation such that the free-field Lagrangian $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2$ is left unchanged. This is called a *fixed point* of the RG flow. Analyzing what happens in the vicinity of the fixed point, we see that the coefficients of $\phi^2$, $\phi^4$, $(\partial_\mu\phi)^4$, $\phi^6$ etc scale respectively as:

$$m'^2 = m^2 b^{-2}, \quad \lambda' = \lambda b^{d-4}, \quad C' = Cb^d, \quad D' = D^{2d-6}, \quad \text{etc.}$$
$$C'_{N,M} = b^{N(d/2-1)+M-d} C_{N,M}, \qquad \text{N powers of } \phi\text{, M derivatives}$$

$b < 1$, so we see that operators multiplying positive powers of $b$ decay (and are called irrelevant), while those multiplying negative powers of $b$ grow (and are called relevant). Operators that are independent of $b$ are called marginal.

So, this tells us that $m^2$ is always relevant, while $\lambda$ is relevant if $d < 4$ (the dimension 4 appears here as special!). More than that, the general relation for an operator $C_{N,M}$ has $(d_{N,M} - d)$ in the exponent of $b$, where $d_{N,M}$ is precisely the mass dimension of the operator (that has to be smaller than $d$ for the theory to be renormalizable). So, relevant operators around $\mathcal{L}_0$ correspond to renormalizable theories! This means that, for example, it is not a coincidence that only renormalizable terms appear in QED: whatever the more fundamental, high-energy Lagrangian could be, when we apply the RG flow to it to go to low energy scales all the non-renormalizable terms go to zero (because they correspond to irrelevant couplings).

For $d > 4$, only the $\phi^2$ coupling is relevant, namely $m^2$. See it in figure 5 (a). For $d = 4$, $\lambda$ is marginal. Looking at the specific transformation law of $\lambda$ under RG flow we find out that it goes to zero, but slower than in the previous case. See it in figure 5 (b). Lastly, for $d < 4$, the operator $\phi^4$ is relevant. However, when $\lambda$ becomes large, we must include nonlinear corrections to the scaling law, that turns out to be:

$$\lambda' = \left[\lambda - \frac{3\lambda^2}{(4\pi)^{d/2}\Gamma(d/2)} \frac{b^{d-4}-1}{4-d}\right] b^{d-4} \tag{22}$$

In this case there is another fixed point, when $\lambda' = \lambda$ solves this equation. For $d \to 4$ it tends to the free-field fixed point, so when d is close to 4 we can compute the transformations equations with perturbation theory, and the new fixed point also has $m^2$ relevant. This is shown in picture 5 (c). We will call it the Wilson-Fisher fixed point.

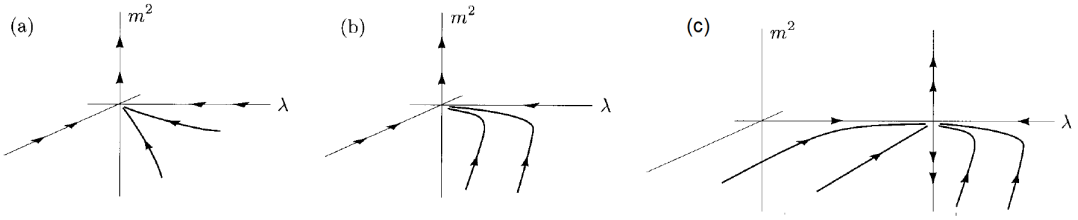
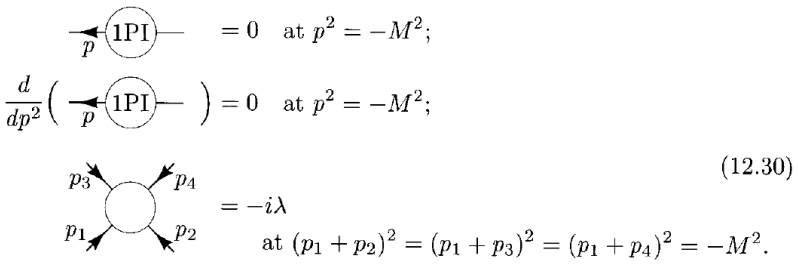



Figure 5: RG flows in the theory space for (a) $d > 4$, (b) $d = 4$, (c) $d < 4$. Figure taken from [4].

## 3.2 Continuum Renormalization Group: The Callan-Symanzik Equation

When we want to extract quantitative predictions, it's technically much easier to work in the continuum renormalization group than to integrate out slices of momentum space like Wilson did (in addition, as we mentioned, the cut-off method is problematic in QFT).




(12.30)

Figure 6: Renormalization conditions. Equations taken from [4].

We construct the continuum RG by looking at how the fields and couplings change in massless $\phi^4$ theory when we make a slight change in the renormalization conditions. Here we are going to work with dimensional regularization and the renormalization conditions in figure 3.2. That are just the on-shell renormalization conditions, but in a point $M$ that is not the physical mass.

These conditions are used to find the counterterm factor $Z_\phi$:

$$\phi_R = Z_\phi^{-1/2}\phi_B \tag{23}$$

When we change the renormalization scale $M$, the renormalized field and the $\phi^4$ coupling suffer correspondent shifts:

$$M \to M + \delta M \qquad \phi_R \to (1 + \delta\eta)\phi_R \qquad \lambda \to \lambda + \delta\lambda$$

Note that all the bare fields remain unchanged. The connected n-point Green's function, being formed of n fields, has the following renormalization rule and change under the shift of $M$, respectively:

$$G_R^{(n)} = Z_\phi^{-n/2} G_B^{(n)} \ ; \ G_R^{(n)} \to (1 + n\delta\eta) G_B^{(n)} \tag{24}$$

If we look at $G_R^{(n)}$ as a function of $M$ and $\lambda$, we can expand $dG_R^{(n)}$ $\left(= n\delta\eta G_R^{(n)}\right)$ in terms of partial derivatives and, defining the quantities

$$\beta \equiv \frac{M}{\delta M}\delta\lambda \qquad \gamma \equiv -\frac{M}{\delta M}\delta\eta \ , \tag{25}$$

we get:

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + n\gamma\right] G_R^{(n)} = 0 \tag{26}$$

this is the *Callan-Symanzik Equation*.

From equation (23), we get

$$Z_\phi^{1/2}(M)\phi_R = Z_\phi^{1/2}(M + \delta M)(1 + \delta\eta)\phi_R$$

$$\Downarrow$$

$$\delta\eta = \frac{Z_\phi(M + \delta M)^{-1/2} - Z_\phi(M)^{-1/2}}{Z_\phi(M)^{-1/2}} \approx \frac{\partial(Z_\phi)^{-1/2}(M)}{\partial M}\frac{\delta M}{Z_\phi(M)^{-1/2}}$$

So, we can rewrite equation (25) as

$$\beta(\lambda) = M\frac{\partial\lambda}{\partial M}\bigg|_{\lambda_B,\Lambda} \qquad \gamma(\lambda) = \frac{1}{2}\frac{M}{Z_\phi}\frac{\partial Z_\phi}{\partial M} \tag{27}$$

$\beta$ and $\gamma$ are functions only of the dimensionless variable $\lambda$. Here, we can see that these functions dictate the running of the coupling constant and the field strength rescaling. In a general theory of massless scalar fields, we would have one $\beta$ for each coupling and one $\gamma$ for each field. Notice that a fixed point of the renormalization group flow in the $\lambda$ direction is a point where $\beta = 0$, and it can be shown that for $d < 4$ this happens at the same Wilson-Fisher fixed point defined before.

Before showing the solution to the Callan-Symanzik equation, let's write down a generalization. Suppose now that our theory is not massless $\phi^4$ anymore, but instead it contains a lot of different local operators $\mathcal{O}_i$ that depend only on the field $\phi$ and on its derivatives. These operators rescale under renormalization as

$$\mathcal{O}_B = Z_{\mathcal{O}}(M)\mathcal{O}_M$$

So, we can do the same procedure as we did to obtain equation (26) in order to include the new operators into the Callan-Symanzik equation, defining the running

$$\gamma_{\mathcal{O}} = M\frac{\partial}{\partial M}\log Z_{\mathcal{O}}(M) \tag{28}$$

If we have a Green's function that contains $n$ powers of $\phi$ and one power of $\mathcal{O}$, $G^{(n,1)} = \langle\phi(p_1)...\phi(p_n)\mathcal{O}_M(k)$, then we follow the same procedure as before and find that the Callan-Symanzik equation for this Green's function is:

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + \gamma_{\mathcal{O}(\lambda)}\right]G_R^{(n,1)} = 0$$

If we had a Green's function that includes $l$ powers of $\mathcal{O}$, we would obtain the equation above with $l$ multiplying the $\gamma_{\mathcal{O}}$ term.

Now, if we add the operator $\mathcal{O}_i$ with coupling $\rho_i$ to our Lagrangian, then the Green's function of $n$ scalar fields $G^{(n)} = \langle\phi(p_1)...\phi(p_n)\rangle$ will be expressed as a perturbation series with all possible powers of the parameter $\rho_i$, and the coefficient of $\rho_i^l$ would be precisely the Green's functions with all the $n$ scalar fields and the $l$ powers of the operator $\mathcal{O}_i$. So, it obeys the Callan-Symanzik equation with $l$ multiplying $\gamma_{\mathcal{O}_i}$. To count the number $l$ in each term of the expansion in $\rho_i$, we use the operator $\rho_i(\partial/\partial\rho_i)$:

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + \sum_i \beta_i\frac{\partial}{\partial\rho_i} + n\gamma\right]G^{(n)}(\{p_i\}, M, \lambda, \{\rho_i\}) = 0$$

$$\beta_i \equiv (d_i - 4 + \gamma_i)\rho_i$$

where the term $(d_i - 4)$ appears because we want $\rho_i$ to be dimensionless.

The solution to this Callan-Symanzik equation can be found by dimensional analysis on $G^{(n)}$. For example, the solution when $n = 2$ will be given in equation (29) of section 4.1.

# 4 Finding the Critical Exponents with the RG

The most important step to connect all the theory of the Renormalization Group with the phase transitions in statistical mechanics is to **identify the critical point with the fixed point of the RG flow**.

Before getting into the details of the method, let's justify this identification by looking at configurations of the two dimensional Ising model (produced by computer simulation) that are shown in [5]. In figure 7, we have a configuration that is slightly above the critical temperature, then we apply on it what we will call a *block-spin transformation*: we group the sites into $3 \times 3$ blocks and assign spins to the blocks according to the "majority rule": if there are more spins up inside it, we assign a spin up to it, and *vice versa*. Finally, we rescale by a factor of 3 so our blocks have the same size as the original squares. What we are doing is to look at the system from a larger scale.

In the initial configuration we had big clusters of correlated spins, but after the transformation the configuration looks more randomized. This happens because we are above the critical temperature: we are in the disordered phase. If we were in the ordered phase ($T < T_C$), the system would choose one orientation and would go closer and closer to the configuration with all the spins aligned. But what happens at the point $T = T_C$? As shown in figure 8, the configuration does not suffer substantial changes in this case. Actually, they are statistically the same: both configurations are equally probable for the Ising Model. This means that, at the critical point of a second-order phase transition, the system has *scale invariance*; fluctuations occur on all length scales, even if we look at arbitrarily large scales. So, the correlation length diverges at this point. A really good demonstration of this scale invariance at the critical point is shown by Douglas Ashton in a 25 seconds long Youtube video [6].

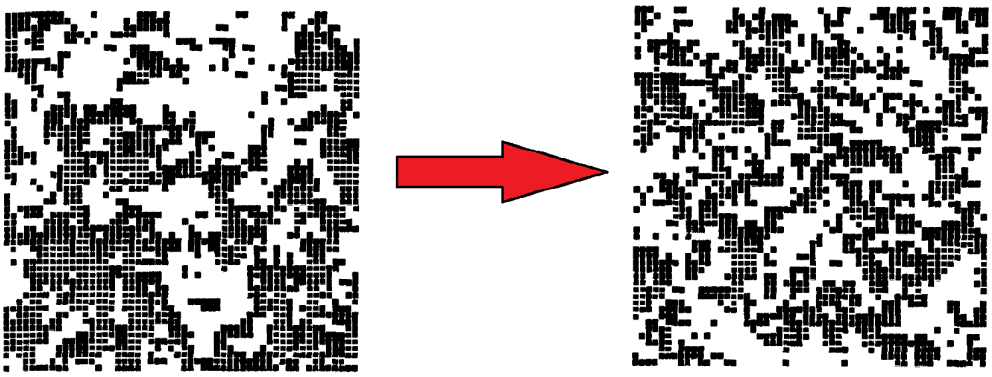


Figure 7: Block spin transformation being applied on a configuration that's slightly above the critical temperature in the 2D Ising Model. Figure taken from [5].

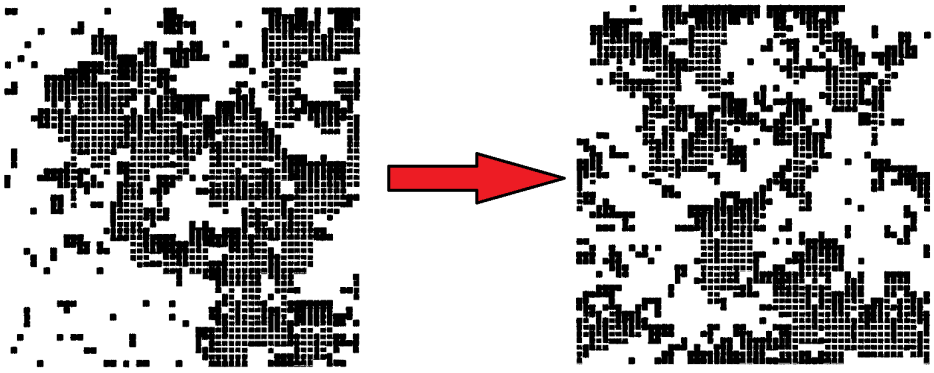


Figure 8: Block spin transformation being applied on a configuration that's at the critical temperature in the 2D Ising Model. Figure taken from [5].

This justifies the identification of the critical point in the Ising Model phase transition with the fixed point of the Renormalization Group: the fixed point is precisely when we have scale invariance, so the RG flow does not affect our theory. When we have more than one fixed point, as in figure 5 (c), the critical point will be the point that has only one relevant direction (in the case, the Wilson-Fisher fixed point). This happens because, if we were at any other fixed point, we could go to the referred one by applying the RG flow more times.

As we saw in section 3.1, near the fixed point all the irrelevant operators go to zero. So, this is in accordance with the idea that the form of the full Lagrangian (the short-distance interactions of the specific system) is not important for the behavior around the critical point: under the view of the Renormalization Group, this explains universality.

Another feature that's immediately explained by this identification is the upper critical dimension $d = 4$: this is the dimension where the Wilson-Fisher fixed point starts coinciding with the free-field fixed point. For mean field theory to work, we need the couplings to be small: this happens only if the critical point is the free-field fixed point. For $d < 4$ the critical point is the Wilson-Fisher fixed point, because $\lambda$ goes from the free-field point to the Wilson-Fisher fixed point under the flow (see figure 5).

Now, we follow [7] and use Callan-Symanzik equation to derive the scaling laws near the critical point when $d < 4$. This development was first made by Brézin, Guillou and Zinn-Justin [8], and is more rigorous than the one first created by Wilson[9][10]. However, we chose to include previously in these notes the discussion of the Wilsonian RG framework because of the illuminating insights it brings us.

As we will see, the first two critical exponents are related to the two-point correlation function, and

the four last ones are related to thermodynamic quantities.

### 4.1 Spin Correlation Function

The Callan-Symanzik equation for the two-point correlation function (n=2) is:

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_i\beta_i\frac{\partial}{\partial\rho_i}+2\gamma(\lambda)\right]G^{(2)}(x,\mu,\{\rho_i\})=0$$

where we started from a general Lagrangian with all possible analytic couplings. The coupling of $\phi^4$ is included in the (dimensionless) $\rho_i$'s as $\rho_\lambda=\lambda-\lambda_*$, where $\lambda_*$ is the Wilson-Fisher fixed point.

The mass dimension of the field is $[\phi]=\frac{d-2}{2}$, because the Lagrangian must be dimensionless. So, the two-point correlation function is written as:

$$G^{(2)}(x)=\frac{1}{|x|^{d-2}}g(\mu|x|,\{\rho_i\})$$

where $g$ is a dimensionless function. Using this, the solution of the Callan-Symanzik equation will be:

$$G^{(2)}(x)=\frac{1}{|x|^{d-2}}h(\{\bar\rho_i\})\exp\left\{-2\int_{1/\mu}^{|x|}d(\log|x'|)\gamma(\bar\rho_\lambda(x'))\right\} \tag{29}$$

$$\frac{d}{d(\log(1/\mu|x|))}\bar\rho_i=\beta_i(\{\rho_j\}) \tag{30}$$

Where $h$ is another dimensionless initial condition.

Near the Wilson-Fisher fixed point, the mass coupling increases as we go to smaller momenta, but all the other dimensionless parameters become small (see figure 5 (c)). Here, we assume that there's a regime where we can take all the couplings $\rho_i$ with $i\neq m$ to zero, but $\rho_m$ is still small enough that we can still consider that we're close to the fixed point.

To calculate the RG flow of the mass coupling, we remember where the quantity $\beta_m$ came from: in equation (3.2) $\beta_m=(d_m-4+\gamma_m)\rho_m$, where $d_m$ is the mass dimension of the $\phi^2$ term. For this term, $d_m=2$. So, integrating equation (30) near the fixed point $\lambda=\lambda_*$, we get:

$$\bar\rho_m=\rho_m(\mu|x|)^{2-\gamma_m(\lambda_*)} \tag{31}$$

Putting it back in equation (29), together with the assumption that all the other $\rho_i$ are zero, we get:

$$\begin{aligned}G^{(2)}(x)&=\frac{1}{|x|^{d-2}}h\left(\rho_m(\mu|x|)^{2-\gamma_m(\lambda_*)}\right)\ \exp\left[-2\gamma(\lambda_*)(\log|x|-\log(1/\mu))\right]\\&=\frac{1}{|x|^{d-2}}h\left(t(\mu|x|)^{2-\gamma_m(\lambda_*)}\right)\frac{1}{(\mu|x|)^{2\gamma(\lambda_*)}}\end{aligned} \tag{32}$$

where we substituted $\rho_m$ for $t=T-T_C$ in reference to the development of section 2, because we can show that in the mean-field free energy $f[m]$, $t$ is the factor that multiplies the term that's quadratic in the field. This makes sense, because in the RG we see that $\rho_m$ is the only parameter that we have to adjust finely (just like we do with the temperature in phase transitions) in order to fall in the fixed point: all the others go there by themselves.

The spin lattice have a characteristic length scale, called the *correlation length* $\xi$, that is defined as the point when the $\rho_m$ term starts to be important in the Lagrangian:

$$\xi\sim x_0 \qquad\qquad \text{where } \bar\rho_m(x_0)=1$$

this correlation length dictates the point where the correlations in the $\phi$ field begin to die away exponentially.

From equation (31), we see that the correlation length diverges at the critical point, and scales as:

$$\xi\sim|t|^{-\nu} \qquad\qquad \text{where } \boxed{\nu=\frac{1}{2-\gamma_m(\lambda_*)}} \tag{33}$$

This is the first critical exponent we are deriving with the Renormalization Group approach. As we will see, all the critical exponents are going to depend only upon two anomalous dimensions, $\gamma(\lambda_*)$ and $\gamma_m(\lambda_*)$. Indeed, at large distances the exponential decay of the correlation function goes with $\exp\{-|x|/\xi\}$, and we can see from the argument of $h$ in equation (32) that the scaling law is really the one given in equation (33).

At small distances, the argument of $h$ goes to zero. So, $G^{(2)}$ goes as a power law, with $-(d-2+2\gamma(\lambda_*))$. We define from here other critical exponent:

$$\boxed{\eta = 2\gamma(\lambda_*)} \tag{34}$$

If the critical exponent were given by simple dimensional analysis, we would have $G^{(2)} \sim 1/|x|^{d-2}$. So, this is a justification for the name "anomalous dimension" given to $\gamma(\lambda_*)$.

## 4.2 Thermodynamic Functions

To obtain the thermodynamic quantities that we can measure, we want to obtain first the equilibrium Gibbs's Free Energy. All the other quantities can be obtained by derivatives and manipulation of this free energy.

The Gibbs's Free Energy is the Legendre transformation of the Helmholtz's Free Energy that we defined back in section 2.3:

$$G_{eq}(T, [m_{\text{eq}}]) = F_{eq}(T, [h]) + \int h(x) m_{\text{eq}}(x) d^d x$$

where we are letting the external field $h$ depend on the position. If we look back at the analogy between QFT and statistical field theory that was established in equations (18) and (19), we can rewrite them as:

$$Z_{\text{SFT}} = \int \mathcal{D}m e^{-\beta(F[m] - \int h(x)m(x)dx)} = e^{-\beta F_{\text{eq}}[h]}$$
$$Z_{\text{QFT}} = \int \mathcal{D}\phi e^{-\frac{1}{\hbar}(S[\phi] + \int J(x)\phi(x)dx)} = e^{-\frac{1}{\hbar}W[J]}$$

where we included source terms. The Legendre transformation of $W[J]$ is the effective action:

$$\Gamma[\phi_{cl}] = W[J] - \int J(x)\phi_{cl}(x)dx$$

So, the Gibbs's Free Energy is exactly the analog to the effective action in QFT. Then, we are going to use the Callan-Symanzik equation for the effective action to study its behavior, and in the end we will switch it to the Gibbs's Free Energy.

The effective action is written in terms of 1PI functions as

$$\Gamma[\phi_{cl}] = i \sum_{n=2}^{\infty} \frac{1}{n!} \int dx_1 ... dx_n \phi_{cl}(x_1) ... \phi_{cl}(x_n) \Gamma^{(n)}(x_1, ..., x_n) \tag{35}$$

And given that the rescaling for the fields is given by $\phi_R = Z_\phi^{-1/2}\phi_B$, the scaling law for the 1PI function is $\Gamma_R^{(n)} = Z_\phi^{(n/2)}\Gamma_B^{(n)}$ (it is a positive power because we need to amputate the external legs). It is the same scaling law that we had for $G^{(n)}$ in equation (24), but with the opposite sign of the power. So, the Callan-Symanzik equation for $\Gamma^{(n)}$ is almost identical to (26):

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} - n\gamma\right]\Gamma_R^{(n)} = 0$$

If we integrate it with $n$ fields $\phi_{cl}$ and sum over all $n$ we get $\Gamma$ by equation (35). The operator that counts $n$ in each term is $\int dx \phi_{cl}(x) \frac{\delta}{\delta\phi_{cl}(x)}$ applied in $\Gamma$. We can choose the case where $\phi_{cl}$ is constant, in wich the Callan-Symanzik equation for $\Gamma$ is:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}-\gamma(\lambda)\phi_{cl}(x)\frac{\partial}{\partial\phi_{cl}(x)}\right]\Gamma_R([\phi_{cl},\mu,\lambda])=0$$

The solution of this equation will be:

$$\Gamma([\phi_{cl}])=\phi^d k(\{\bar{\rho}_i(\phi_{cl})\})\,\exp\left[-\int_{\mu^{(d-2)/2}}^{\phi_{cl}} d(\log|\phi'|)\frac{4\gamma d}{(d-2)(d-2+2\gamma)}(\bar{\rho}(\phi'))\right] \tag{36}$$

$$\frac{d}{d(\log(\phi_{cl}))}\bar{\rho}_i=\frac{2\beta_i(\{\rho_j\})}{d-2+2\gamma(\{\rho_j\})} \tag{37}$$

Where k is a dimensionless initial condition to be determined by perturbation theory. Again, we specialize to a region close to the Wilson-Fisher fixed point:

$$\frac{d}{d(\log(\phi_{cl}))}\bar{\rho}_m=\frac{2(-2+\gamma_m(\lambda_*))}{d-2+2\gamma(\lambda_*)}\bar{\rho}_m$$

$$\Downarrow$$

$$\bar{\rho}_m=\rho_m(\phi_{cl}\mu^{-(d-2)/2})^{-2(2-\gamma_m(\lambda_*))/(d-2+2\gamma(\lambda_*))}$$

Ignore irrelevant operators: $\bar{\rho}_i=0$ for $i\neq m$. To get the Gibbs free energy instead of the effective action $\Gamma$, we simply change $\rho_m\to t$ and $\phi_{cl}\to M$ (the magnetization).

In the approximation we just made (near the fixed point), the Gibbs Free Energy is:

$$G(M,t)=M^d k\left(t(M\mu^{-(d-2)/2})^{-2(2-\gamma_m(\lambda_*))/(d-2+2\gamma(\lambda_*))}\right)\left(\frac{\mu^{(d-2)/2}}{M}\right)^{\frac{4d\gamma(\lambda_*)}{(d-2)(d-2+2\gamma(\lambda_*))}} \tag{38}$$

From this equation, we can derive the critical exponents $\alpha$, $\beta$, $\gamma$ and $\delta$ (do not confuse the exponents $\beta$ and $\gamma$ with the runnings):

- Setting $t=0$ (we sit at the critical temperature), we do the derivative of the free energy with respect to the field $M$, which gives us the field $H$. Inverting the resulting expression, we find $\delta$.
- When we set $H=0$ (to look at the second-order phase transition), we are finding the minimum of $G$ with respect to $M$. This minimum is nonzero in the ordered phase (temperature below $T_C$). If we rewrite $G$ in equation (38) as a power of $t$ times a function $\hat{f}$ of $M$ and $t$, the minimum of $G$ will happen at the minimum of $\hat{f}$. So, when we look at phase transitions with $H=0$, the $\beta$ exponent can be found by saying that the argument of $\hat{f}$ is constant.
- When we are above $T_C$, the minimum of $G$ with respect to $M$ happens at the point $M=0$, so the argument of $\hat{f}$ is zero. So, the free energy will scale just as the power of $t$ that multiplies $\hat{f}$ in the expression for $G$; as it turns out, this power is $t^{\beta(1+\delta)}$. Derivating $G$ with respect to $t$ twice, we find the heat capacity and the critical exponent $\alpha=2-\beta(1+\delta)$.
- Finally, to find the exponent $\gamma$, we notice that $H=\frac{\partial G}{\partial M}=t^{\beta\delta}\hat{f}'(Mt^{-\beta})$, so we can invert this equation and find $M=t^{\beta}\hat{c}(Ht^{-\beta\delta})$, where $\hat{c}$ is another function. Deriving this expression with respect to $H$ we find the magnetic susceptibility, and the critical exponent $\gamma=(\delta-1)\beta$.

The results of all this calculations we just described are summarized below.

$$\boxed{\beta=\frac{d-2+2\gamma(\lambda_*)}{2(2-\gamma_m(\lambda_*))}}\qquad\boxed{\delta=\frac{d+2-2\gamma(\lambda_*)}{d+2+2\gamma(\lambda_*)}}\qquad\boxed{\gamma=\frac{2(1-\gamma(\lambda_*))}{2-\gamma_m(\lambda_*)}}\qquad\boxed{\alpha=2-\frac{2d}{2-\gamma_m(\lambda_*)}} \tag{39}$$

So, the equations (33), (34) and (39) indeed say how to relate all the critical exponents with only two anomalous dimensions, as we mentioned earlier.

It is possible to calculate these quantities by writing down the perturbation theory results for the counterterms and expanding equations (27) and (28) to find the functions $\gamma$ and $\gamma_m$ to first orders. Doing

this procedure to the $\beta$ function, we can find its non-trivial zero $\lambda_*$, and use it to calculate the anomalous dimensions $\gamma(\lambda_*)$ and $\gamma_m(\lambda_*)$.

In this discussion, we've been considering systems that are described by only one scalar field $\phi$. This is true for the example we gave in section 2, where the spins have a preferred axis. However, if we had more than one spin component, for example spins with a preferred plane ($N = 2$) or isotropic spins ($N = 3$), they would be described by 2 and 3 scalar fields, respectively. In this case, the relation between the critical exponents and the anomalous dimensions remains the same, but the results for the anomalous dimensions change. They must be computed in symmetric $O(N)$ $\phi^4$ theory.

In the last page, we show a table of critical exponents taken from [7]. In the column "Landau", we see the results from Mean-Field Approximation and simple dimensional analysis. On the other hand, the column "QFT" shows the results obtained with the techniques described in this section. The dimension is 3. We can see that the RG results are in much more agreement with experiment than the Mean-Field results. More than that, we see the comparison of the critical exponents obtained for different systems, as a demonstration of universality.

# 5 Conclusions

In these notes we introduced the use of the Renormalization Group as a powerful tool to study the behavior of thermodynamical systems near phase transitions. With the renormalization group ideas, we were able to understand the phenomenon of universality (why different systems look the same near second-order phase transitions), and we explained the upper critical dimension $d = 4$ for the Mean-Field approximation of the Ising Model. More than that, we were able to calculate the critical exponents, numbers that dictate the scaling behavior of correlations and thermodynamic quantities near the critical point, with much better accuracy than mean field approximation when $d \leq 4$.

The use of these tools that we learned in QFT was possible because of the analogy between Quantum Field Theory and Statistical Field Theory that is made explicit by the path integral formulation.

# References


[1] David Tong. *Lectures on Statistical Field Theory*. 2017. URL: http://www.damtp.cam.ac.uk/user/tong/sft.html.

[2] Uwe C. Täuber. "Equilibrium critical phenomena". In: *Critical Dynamics: A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior*. Cambridge University Press, 2014, pp. 5–44. DOI: 10.1017/CBO9781139046213.003.

[3] Gabriel Landi. *Statistical Mechanics 2018 - Lecture Notes*. 2018. URL: http://www.fmt.if.usp.br/~gtlandi/courses/statmech2018.html.

[4] Daniel V. Schroeder Michael E. Peskin. "An Introduction to Quantum Field Theory". In: Addinson-Wesley Publishing Company, 1995. Chap. 12.

[5] John Cardy. "Scaling and Renormalizaion in Statistical Physics". In: Cambridge University Press, 1996. Chap. 3.

[6] Douglas Ashton. *Scale invariance in the critical Ising model*. 2011. URL: https://www.youtube.com/watch?v=fi-g2ET97W8.

[7] Daniel V. Schroeder Michael E. Peskin. "An Introduction to Quantum Field Theory". In: Addinson-Wesley Publishing Company, 1995. Chap. 13.

[8] J. Zinn-Justin E. Brezin J. C. Guillou. "Wilson's Theory of Critical Phenomena ans Callan-Symanzik Equations in 4-$\epsilon$ dimensions". In: *Physical Review D* 8.2 (July 1973), pp. 434–440.

[9] Michael E. Fisher Kenneth G. Wilson. "Critical Exponents in 3.99 Dimensions". In: *Physical Review Letters* 28.4 (Jan. 1972), pp. 240–243.

[10] Kenneth G. Wilson. "Feynman-Graph Expansion for Critical Exponents". In: *Phys. Rev. Lett.* 28 (9 Feb. 1972), pp. 548–551. DOI: 10.1103/PhysRevLett.28.548. URL: https://link.aps.org/doi/10.1103/PhysRevLett.28.548.

**Table 13.1.** Values of Critical Exponents for Three-Dimensional Statistical Systems

| Exponent | Landau | QFT | Lattice | Experiment | |
|---|---|---|---|---|---|
| $N = 1$ Systems: | | | | | |
| $\gamma$ | 1.0 | 1.241 (2) | 1.239 (3) | 1.240 (7) | binary liquid |
| | | | | 1.22 (3) | liquid-gas |
| | | | | 1.24 (2) | $\beta$-brass |
| $\nu$ | 0.5 | 0.630 (2) | 0.631 (3) | 0.625 (5) | binary liquid |
| | | | | 0.65 (2) | $\beta$-brass |
| $\alpha$ | 0.0 | 0.110 (5) | 0.103 (6) | 0.113 (5) | binary liquid |
| | | | | 0.12 (2) | liquid-gas |
| $\beta$ | 0.5 | 0.325 (2) | 0.329 (9) | 0.325 (5) | binary liquid |
| | | | | 0.34 (1) | liquid-gas |
| $\eta$ | 0.0 | 0.032 (3) | 0.027(5) | 0.016 (7) | binary liquid |
| | | | | 0.04 (2) | $\beta$-brass |
| $N = 2$ Systems: | | | | | |
| $\gamma$ | 1.0 | 1.316 (3) | 1.32 (1) | | |
| $\nu$ | 0.5 | 0.670 (3) | 0.674 (6) | 0.672 (1) | superfluid $^4$He |
| $\alpha$ | 0.0 | −0.007 (6) | 0.01 (3) | −0.013 (3) | superfluid $^4$He |
| $N = 3$ Systems: | | | | | |
| $\gamma$ | 1.0 | 1.386 (4) | 1.40 (3) | 1.40 (3) | EuO, EuS |
| | | | | 1.33 (3) | Ni |
| | | | | 1.40 (3) | $RbMnF_3$ |
| $\nu$ | 0.5 | 0.705 (3) | 0.711 (8) | 0.70 (2) | EuO, EuS |
| | | | | 0.724 (8) | $RbMnF_3$ |
| $\alpha$ | 0.0 | −0.115 (9) | −0.09 (6) | −0.011 (2) | Ni |
| $\beta$ | 0.5 | 0.365 (3) | 0.37 (5) | 0.37 (2) | EuO, EuS |
| | | | | 0.348 (5) | Ni |
| | | | | 0.316 (8) | $RbMnF_3$ |
| $\eta$ | 0.0 | 0.033 (4) | 0.041 (14) | | |

The values of critical exponents in the column 'QFT' are obtained by resumming the perturbation series for anomalous dimensions at the Wilson-Fisher fixed point in $O(N)$-symmetric $\phi^4$ theory in three dimensions. The values in the column 'Lattice' are based on analysis of high-temperature series expansions for lattice statistical mechanical models. The values in the column 'Experiment' are taken from experiments on critical points in the systems described. In all cases, the numbers in parentheses are the standard errors in the last displayed digits. This table is based on J. C. Le Guillou and J. Zinn-Justin, *Phys. Rev.* **B21**, 3976 (1980), with some values updated from J. Zinn-Justin (1993), Chapter 27. A full set of references for the last two columns can be found in these sources.