

THE SPHERICAL MODEL

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1. THE SPHERICAL MODEL

We will consider a variant of the Ising model where the spins σ_x living on a hypercubic lattice can take on any values in the real numbers. The integral of the Boltzmann weight

$$e^{\frac{J}{k_B T} \sum_{x,i} \sigma_x \sigma_{x+i} + \frac{B}{k_B T} \sum_x \sigma_x}$$

over the spins σ_x would be badly divergent. This divergence is absent in the Ising model because the magnitude of the spins is limited. Something like that would be needed in order to make the integral converge. The spherical model uses this Boltzmann weight, but it puts a strict limit on the sum over all of the spins,

$$\sum_x \sigma_x^2 = N$$

where N is the total number of spins. Given this limitation, no spin can have arbitrarily large magnitude but it can still have positive or negative signs.

In fact, if we put the external field to zero, $B = 0$, this model has Z_2 symmetry like the Ising model, where the Z_2 transformation flips the sign of every spin $Z_2 : \sigma_x \rightarrow -\sigma_x$.

This model has a ferromagnetic ground state. To see this, consider the Hamiltonian

$$H = -J \sum_{x,i} \sigma_x \sigma_{x+i} - B \sum_x \sigma_x + \lambda \left(\sum_x \sigma_x^2 - N \right)$$

where λ is a Lagrange multiplier enforcing the constraint. The equations for its extrema are

$$-J \sum_i (\sigma_{x+i} + \sigma_{x-i}) - B + 2\lambda \sigma_x = 0, \quad \sum_x \sigma_x^2 = N$$

These equations are solved by

$$\sigma_x = \text{sign}(B), \quad \lambda = DJ + \frac{|B|}{2}$$

We can easily see by inspection that, once the constraint is imposed, this solution is an absolute minimum of each term in the Hamiltonian, so it must be the minimum energy solution. It is indeed Ferromagnetic with a magnetization density $m = \text{sign}(B)$.

The internal energy of this state is obtained by putting the solution into the Hamiltonian to get

$$U = -N(2DJ + |B|)$$

which is non-analytic at $B = 0$ where there is a first order phase transition between two ferromagnetic states, one with $m = 1$ and the other with $m = -1$. We will see in the following that this critical point is the terminus of a line of first order phase transitions that lies on the T -axis in the (B, T) -plane. The line of phase transitions will terminate at a critical point $(0, T_C)$ where there is a second order phase transition. It will be the immediate vicinity of that second order transition which will be of great interest to us. The phase diagram is depicted in figure 1.

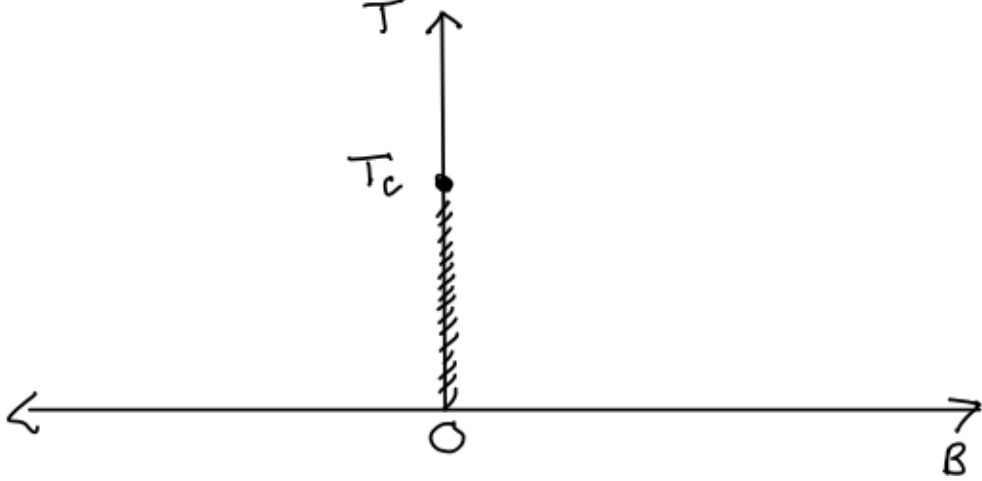


FIGURE 1. The phase diagram of the spherical model is depicted. There is a line of first order phase transitions located on the segment between zero and T_C of the $B = 0$ axis, denoted by the crosshatch segment, and ending at a second order phase transition at the critical point $(T, B) = (T_C, 0)$.

We can then form the partition function of the spherical model as sum – in this case integral – over all of the configurations of the spins in the system,

$$(1) \quad Z[T, B, N] = \int_{-\infty}^{\infty} \prod_x d\sigma_x e^{\frac{J}{k_B T} \sum_{x,i} \sigma_x \sigma_{x+i} + \frac{B}{k_B T} \sum_x \sigma_x} \delta\left(\sum_x \sigma_x^2 - N\right)$$

In order to impose the constraint we use the Fourier transform of the Dirac delta function

$$\delta\left(\sum_x \sigma_x^2 - N\right) = \int \frac{d\lambda}{2\pi} e^{i\lambda(\sum_x \sigma_x^2 - N)}$$

to write the partition function as

$$(2) \quad Z[T, B, N] = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int_{-\infty}^{\infty} \prod_x d\sigma_x e^{\frac{J}{k_B T} \sum_{x,i} \sigma_x \sigma_{x+i} + \frac{B}{k_B T} \sum_x \sigma_x + i\lambda \sum_x (\sigma_x^2 - N)}$$

The integration over σ_x is now a Gaussian integration. It has the form

$$(3) \quad Z[T, B, N] = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-iN\lambda} \int_{-\infty}^{\infty} \prod_x d\sigma_x e^{-\frac{1}{2} \sum_{x,y} \sigma_x \Delta(x,y) \sigma_y + \frac{B}{k_B T} \sum_x \sigma_x}$$

$$(4) \quad = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-iN\lambda} e^{\frac{B^2}{2(k_B T)^2} \sum_{xy} \Delta^{-1}(x,y) - \frac{1}{2} \text{Tr} \ln[\Delta/2\pi]}$$

where

$$(5) \quad \Delta(x, y) = \frac{J}{k_B T} \sum_i \left[2 - \delta_{y, x+\hat{i}} - \delta_{y, x-\hat{i}} \right] + 2(-i\lambda - \frac{JD}{k_B T}) \delta_{xy}$$

The eigenvalues of $\Delta(x, y)$ are (see the appendix for more discussion)

$$(6) \quad \Delta(p) = \frac{J}{k_B T} \sum_i 4 \sin^2 \frac{p_i}{2} + 2(-i\lambda - \frac{JD}{k_B T})$$

and the partition function becomes

$$(7) \quad Z[T, B, N] = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-iN\lambda} e^{N \frac{B^2}{2(k_B T)^2} \frac{1}{2(-i\lambda - \frac{JD}{k_B T})} - \frac{N}{2} \int \frac{d^D p}{(2\pi)^D} \ln[\frac{J}{k_B T} \sum_i 4 \sin^2 \frac{p_i}{2} + 2(-i\lambda - \frac{JD}{k_B T})]}$$

It remains to do the integral over λ . Since N is very large, this integral can be done by saddle point technique. The result, for the free energy, is

$$(8) \quad F[T, B, N] = k_B T N \inf_{\lambda} \left\{ -i\lambda - \frac{B^2}{2(k_B T)^2} \frac{1}{2(-i\lambda - \frac{JD}{k_B T})} + \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \ln[\frac{J}{k_B T} \sum_i 4 \sin^2 \frac{p_i}{2} + 2(-i\lambda - \frac{JD}{k_B T})] \right\}$$

The equation which determines the saddle point is

$$(9) \quad 1 - \frac{B^2}{(k_B T)^2} \frac{1}{\kappa^4} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\frac{J}{k_B T} \sum_i 4 \sin^2 \frac{p_i}{2} + \kappa^2}$$

$$(10) \quad \kappa^2 = 2 \left(-i\lambda - \frac{JD}{k_B T} \right)$$

In principle, we look for solutions of this equation in the entire complex plane. It will turn out that it has a saddle point at a complex value of λ , such that $i\lambda$ is a real number and such that κ^2 that is defined in the equation above is a non-negative real number.

To get some intuition for how this comes about, we can consider the high temperature limit where $k_B T \gg B$ and $k_B T \gg J$. In that limit, the equation for λ , (9) reduces to

$$1 = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\kappa^2} = \frac{1}{\kappa^2}$$

and $\kappa^2 \rightarrow 1$ in this limit.

Now, let us lower the temperature. For the moment, let us set $B = 0$. Let us assume that there is no spontaneous magnetization. Then, from equation (9) we can see that

$$B = 0 : \frac{\partial \kappa^2}{\partial T} > 0$$

This means that, as we decrease T , κ^2 must also decrease. Note that the above equation has a strict inequality. This means that κ^2 does not have a fixed point where it stops decreasing, at least in the regime where our assumptions that $B = 0$ and there is no spontaneous magnetization are valid.

At very large temperatures, κ^2 is equal to one. As we lower the temperature, κ^2 must decrease. The critical point occurs where κ^2 decreases to zero. This happens at a critical temperature, T_C , and we can deduce this critical temperature from equation (9) as

$$(11) \quad \frac{J}{k_B T_C} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\sum_i 4 \sin^2 \frac{p_i}{2}}$$

The integral on the right-hand-side of this equation converges when $D > 2$. When $D = 2$ it is logarithmically divergent with the divergence coming from the regime of small values of the p_i . We interpret this as

$$T_C = 0 \text{ when } D \leq 2$$

When $D > 2$ we can rewrite equation (9) as

$$(12) \quad \left(\frac{J}{k_B T} \right)^2 \left(\frac{T}{T_C} - 1 \right) = \kappa^2 \int \frac{d^D p}{(2\pi)^D} \frac{1}{[\sum_j 4 \sin^2 \frac{p_j}{2}][\sum_i 4 \sin^2 \frac{p_i}{2} + \frac{k_B T}{J} \kappa^2]}$$

The behaviour of the right-hand-side of the above equation as $\kappa^2 \rightarrow 0$ depends on the dimension. If $D > 4$ the integral over wave-numbers is convergent even when the κ^2 in the denominator vanishes. In that case, equation (12) implies

$$D > 4 : \kappa^2 \sim \left(\frac{T}{T_C} - 1 \right) \text{ as } T \rightarrow T_C$$

As we shall see shortly, this equation will determine some of the critical exponents of the spherical model.

When $2 < D \leq 4$ we need to be more sophisticated about how we take the limit where κ is small in the right-hand-side of equation (12). The reason for this is the fact that, when $D < 4$ and when $\kappa \rightarrow 0$, the integral on the right-hand-side of equation (12) diverges. The divergence comes from the regime where $p_i \sim 0$ (and this is called an infrared divergence). To focus in on it, we can split the integration into two regimes, one over a small sphere of radius $\Lambda \ll \pi$ and centered on the origin and another regime which is the remainder of the Brillouin zone. We denote the integral over the small sphere as $\int_{p < \Lambda}$ and the integral

over the rest of the zone as $\int_{p>\Lambda}$. Then, with this distinction

$$(13) \quad \left(\frac{J}{k_B T}\right)^2 \left(\frac{T}{T_C} - 1\right) = \kappa^2 \int_{p<\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{[\sum_j 4 \sin^2 \frac{p_j}{2}][\sum_i 4 \sin^2 \frac{p_i}{2} + \frac{k_B T}{J} \kappa^2]} \\ + \kappa^2 \int_{p>\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{[\sum_j 4 \sin^2 \frac{p_j}{2}][\sum_i 4 \sin^2 \frac{p_i}{2} + \frac{k_B T}{J} \kappa^2]}$$

Then, the last term on the right-hand-side of equation (14) has a smooth limit as $\kappa \rightarrow 0$ and in the regime where κ is very small it is $\sim \kappa^2$. The first term does not have a smooth limit and it requires further study. Let us assume that the “cutoff” Λ is so small that we can take

$$\lim_{\kappa^2 \rightarrow 0} \kappa^2 \int_{p<\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{[\sum_j 4 \sin^2 \frac{p_j}{2}][\sum_i 4 \sin^2 \frac{p_i}{2} + \frac{k_B T}{J} \kappa^2]} = \lim_{\kappa^2 \rightarrow 0} \kappa^2 \int_{p<\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{[\vec{p}^2][\vec{p}^2 + \frac{k_B T}{J} \kappa^2]} \\ = \lim_{\kappa^2 \rightarrow 0} \kappa^2 \int_{R^D} \frac{d^D p}{(2\pi)^D} \frac{1}{[\vec{p}^2][\vec{p}^2 + \frac{k_B T}{J} \kappa^2]} - \lim_{\kappa^2 \rightarrow 0} \kappa^2 \int_{p>\Lambda, p \in \Omega_B} \frac{d^D p}{(2\pi)^D} \frac{1}{[\vec{p}^2][\vec{p}^2 + \frac{k_B T}{J} \kappa^2]} \\ = \lim_{\kappa^2 \rightarrow 0} \left[\frac{k_B T}{J} \kappa^2\right]^{\frac{D-2}{2}} \int_{R^D} \frac{d^D p}{(2\pi)^D} \frac{1}{[\vec{p}^2][\vec{p}^2 + 1]} - \lim_{\kappa^2 \rightarrow 0} \kappa^2 \int_{p>\Lambda, p \in \Omega_B} \frac{d^D p}{(2\pi)^D} \frac{1}{[\vec{p}^2][\vec{p}^2]}$$

Then the result for our equation which determines κ^2 is

$$(14) \quad \lim_{T \rightarrow T_C} \left(\frac{J}{k_B T}\right)^2 \left(\frac{T}{T_C} - 1\right) = \lim_{\kappa^2 \rightarrow 0} \left[\frac{k_B T}{J} \kappa^2\right]^{\frac{D-2}{2}} \int_{R^D} \frac{d^D p}{(2\pi)^D} \frac{1}{[\vec{p}^2][\vec{p}^2 + 1]} \\ - \lim_{\kappa^2 \rightarrow 0} \kappa^2 \int_{p>\Lambda, p \in \Omega_B} \frac{d^D p}{(2\pi)^D} \frac{1}{[\vec{p}^2][\vec{p}^2]} + \kappa^2 \int_{p>\Lambda, p \in \Omega_B} \frac{d^D p}{(2\pi)^D} \frac{1}{[\sum_j 4 \sin^2 \frac{p_j}{2}][\sum_i 4 \sin^2 \frac{p_i}{2}]}$$

The last two terms vary like κ^2 for small κ whereas the first term on the right-hand-side varies like $[\kappa]^{D-2}$ which is dominant as $\kappa \rightarrow 0$. We get the scaling law

$$(15) \quad \kappa^2 \sim (T - T_C)^{\frac{2}{D-2}} \quad T \rightarrow T_C^+, \quad 2 < D < 4$$

In summary, the equation for κ has the form

$$(16) \quad \begin{array}{lll} D \leq 2 & T_C = 0 & \kappa^2 > 0 \\ 2 < D < 4 & \left(\frac{J}{k_B T_C}\right)^{\frac{D}{2}} \left(\frac{T}{T_C} - 1\right) = \kappa^{D-2} \left[\frac{-\Gamma[1-D/2]}{(4\pi)^{\frac{D}{2}}}\right] + \mathcal{O}(\kappa^2) & \kappa^2 \sim \left(\frac{T}{T_C} - 1\right)^{\frac{2}{D-2}} \\ D = 4 & \left(\frac{J}{k_B T_C}\right)^2 \left(\frac{T}{T_C} - 1\right) = 2\pi^2 \kappa^2 \ln \frac{\Lambda}{\kappa} + \mathcal{O}(\kappa^2) & \\ D > 4 & \left(\frac{J}{k_B T_C}\right)^2 \left(\frac{T}{T_C} - 1\right) = \kappa^2 \left(\frac{J}{k_B T_C}\right)^2 & \kappa^2 \sim \left(\frac{T}{T_C} - 1\right) \end{array}$$

We note that $D = 4$ is special and we will find some reasons for this in our later work on the renormalization group and the Ising model. Indeed, the scaling law is corrected by logarithms as we have shown in the above equation.

To see how the scaling of κ^2 translates to critical behaviour, it is best to examine the Helmholtz free energy in the $T > T_C$ regime.

(17)

$$F[T, B, N] = k_B T N \left\{ \frac{\kappa^2}{2} + \frac{JD}{k_B T} - \frac{B^2}{2(k_B T)^2} \frac{1}{\kappa^2} + \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \ln \left[\frac{J}{k_B T} \sum_i 4 \sin^2 \frac{p_i}{2} + \kappa^2 \right] \right\}$$

It is easy to see that any nonanalyticity of this free energy will come from the variation of κ^2 near $\kappa^2 = 0$.

The magnetization is defined as

$$m \equiv \frac{M}{N} = -\frac{1}{N} \frac{\partial}{\partial B} F[T, B, N]_{T, N} = \frac{B}{k_B T \kappa^2}$$

and the magnetic susceptibility is given by

$$\chi = \frac{\partial m}{\partial B} = \frac{1}{k_B T \kappa^2} - \frac{B}{k_B T} \frac{\partial \kappa^2}{\partial B}$$

At vanishing field

$$\lim_{B \rightarrow 0} \chi = \frac{1}{k_B T \kappa^2} \sim \begin{cases} \left(\frac{T}{T_C} - 1 \right)^{-\frac{2}{D-2}} & 2 < D < 4 \\ \left(\frac{T}{T_C} - 1 \right)^{-1} & D > 4 \end{cases}$$

Below the critical temperature $\kappa^2 = 0$ and there must be a magnetization. Returning to the equation for κ^2 with $B = k_B T \kappa^2 m$ restored and κ^2 subsequently set to zero, we get

$$(18) \quad 1 - m^2 = \int \frac{d^D p}{(2\pi)^D} \frac{1}{\frac{J}{k_B T} \sum_i 4 \sin^2 \frac{p_i}{2}} = \frac{T}{T_C}$$

and we find a very simple equation for the spontaneous magnetization

$$(19) \quad m \sim \left(1 - \frac{T}{T_C} \right)^{-\frac{1}{2}}$$

2. $B \neq 0$ AND $T = T_C$

Now let us allow B to be nonzero but we put $T = T_C$ and approach the critical point along a horizontal direction in figure 1. The equation that we need to solve for κ is

$$(20) \quad 1 - \frac{B^2}{2J^2} \frac{1}{\kappa^4} = \frac{k_B T_C}{J} \int \frac{d^D p}{(2\pi)^D} \frac{1}{\left[\sum_i 4 \sin^2 p_i / 2 + \kappa^2 \right]} = \frac{\int \frac{d^D p}{(2\pi)^D} \frac{1}{\left[\sum_i 4 \sin^2 p_i / 2 + \kappa^2 \right]}}{\int \frac{d^D p}{(2\pi)^D} \frac{1}{\left[\sum_i 4 \sin^2 p_i / 2 \right]}}$$

in the limit where κ is very small. When $2 < D < 4$, we use the same technique of rescaling $p_i \rightarrow \kappa p_i$ that we have already used, we see that this equation becomes

$$(21) \quad 1 - \frac{B^2}{2J^2} \frac{1}{\kappa^4} = 1 - \frac{k_B T_C}{J} [\kappa^2]^{\frac{D}{2}-1} \int \frac{d^D \kappa}{(2\pi)^4} \frac{1}{k^2(k^2 + 1)}$$

When $D > 4$ we can simply expand the right-hand-side so that, there

$$(22) \quad 1 - \frac{B^2}{2J^2} \frac{1}{\kappa^4} \sim 1 - \kappa^2$$

Equations (21) and (22) tell us that

$$(23) \quad \kappa^2 \sim B^{\frac{4}{D+2}}, \quad 2 < D < 4$$

$$(24) \quad \kappa^2 \sim B^{\frac{2}{3}}, \quad D > 4$$

which then implies that

$$(25) \quad m \sim \frac{B}{\kappa^2} \sim B^{\frac{D-2}{D+2}}, \quad 2 < D < 4$$

$$(26) \quad m \sim B^{\frac{1}{3}}, \quad D > 4$$

3. FREE ENERGY WITH $B = 0$

Finally, the free energy at zero field, $B = 0$, as a function of T , has a singularity at $T = T_C$. In the high temperature regime $T > T_C$, and with $T \sim T_C$ it is given, for small κ^2 ,

$$(27) \quad \hat{F}[T, N, 0] = \frac{Nk_B T}{2} \int \frac{d^D p}{(2\pi)^D} \ln \left[\frac{k_B T}{2\pi J} \sum_i 4 \sin^2 p_i / 2 \right] + \frac{NJ}{2} \left(\frac{T}{T_C} - 1 \right) \kappa^2 + \dots$$

where we have used (??). We can then use (??) to get

$$T > T_C$$

$$(28)$$

when $2 < D < 4$

$$\hat{F}[T, N, 0] = \frac{Nk_B T}{2} \int \frac{d^D p}{(2\pi)^D} \ln \left[\frac{k_B T}{2\pi J} \sum_i 4 \sin^2 p_i / 2 \right] + \text{const.} \left(\frac{T}{T_C} - 1 \right)^{\frac{D}{D-2}} + \dots,$$

$$(29)$$

when $D > 4$

$$\hat{F}[T, N, 0] = \frac{Nk_B T}{2} \int \frac{d^D p}{(2\pi)^D} \ln \left[\frac{k_B T}{2\pi J} \sum_i 4 \sin^2 p_i / 2 \right] + \text{const.} \left(\frac{T}{T_C} - 1 \right)^2 + \dots,$$

The specific heat is gotten by taking two derivatives

$$C = -T \frac{\partial^2}{\partial T^2} \hat{F} \Big|_{N, B}$$

and the critical exponent α can be deduced from this expression.

4. CRITICAL EXPONENTS

We can compare the critical exponents of the spherical model to those of mean field theory which we found for the infinite range Ising model. They are indeed different when the dimension is in the interval $2 < D < 4$ and they are identical when $D > 4$. Notice that, the exponents in the regime $2 < D < 4$ approach the $D > 4$ exponent at $D = 4$ so that, even though we have not examined this case in detail, we much conjecture that the mean field theory exponents are correct for $D = 4$.

(30)

Infinite range Ising	α	β	γ	δ
	0	$\frac{1}{2}$	1	3
Spherical model	$\left\{ \begin{array}{cc} \frac{D-4}{D-2} & 2 < D < 4 \\ 0 & D > 4 \end{array} \right\}$	$\frac{1}{2}$	$\left\{ \begin{array}{cc} \frac{2}{D-2} & 2 < D < 4 \\ 1 & D > 4 \end{array} \right\}$	$\left\{ \begin{array}{cc} \frac{D+2}{D-2} & 2 < D < 4 \\ 3 & D > 4 \end{array} \right\}$

where the exponents govern the critical behaviour of the following quantities

$$(31) \quad \text{specific heat } C \sim \left(\frac{T}{T_C} - 1 \right)^{-\alpha}, \quad T > T_C$$

$$(32) \quad \text{spontaneous magnetization } m \sim \left(1 - \frac{T}{T_C} \right)^{\beta}, \quad T < T_C$$

$$(33) \quad \text{magnetic susceptibility } \chi \sim \left(\frac{T}{T_C} - 1 \right)^{-\gamma}, \quad T > T_C$$

$$(34) \quad \text{field vs } M \quad B \sim M^{\delta}, \quad T = T_C$$

APPENDIX A. LATTICE FOURIER TRANSFORM

We shall assume that the spins in our spin model reside on the sites of a D -dimensional hypercubic lattice whose points are located in D -dimensional Euclidean space by the formula

$$(35) \quad x = a(n_1 \hat{i}_1 + n_2 \hat{i}_2 + \dots + n_D \hat{i}_D)$$

where $(n_1, n_2, \dots, n_D)$ are a set of D integers and a is the lattice constant, the distance between neighbouring sites. For all possible values of the integers $(n_1, n_2, \dots, n_D)$, x sweeps over the sites of the lattice. We assume that the lattice is infinite, or at least very large with N sites, therefore N unit cells and its volume Na^D .

A plane wave on the lattice is given by the exponential

$$e^{ipx}$$

where $p = (p_1, p_2, \dots, p_D)$ is a vector whose components have the dimensions of inverse length.

Because of the discreteness of the lattice, the plane wave with the wave-vector p and the plane wave with the wave-vector $p + 2\pi\hat{i}/a$, for any of the D unit vectors $\hat{i}$, are identical. In order to enumerate all of the truly distinct plane waves, we must restrict the wave-vector

variables p to a unit cell in the space of wave-vectors which is called the “Brillouin zone”. This unit cell is the region

$$\Omega_B \equiv \{(p_1, p_2, \dots, p_D) \mid -\pi/a < p_i \leq \pi/a\}$$

or, alternatively, the hypertorus which is gotten by the periodic identification of points

$$\Omega_B = \{p : p \sim p + \frac{2\pi}{a}\hat{i}, \forall i = 1, 2, \dots, D\}$$

The lattice delta function has a Fourier integral representation that is given by

$$\delta_{x,y} = \int_{\Omega_B} \frac{d^D p}{(2\pi/a)^D} e^{ip(x-y)}$$

and for the wave-vector delta function

$$\delta^D(p-q) = \frac{1}{(2\pi/a)^D} \sum_{n_1, \dots, n_D = -\infty}^{\infty} e^{i(p-q)(n_1\hat{i}_1 + n_2\hat{i}_2 + \dots + n_D\hat{i}_D)}$$

A function $f(x)$ which is defined on the lattice can be written as a superposition of lattice plane waves as

$$f(x) = \int_{\Omega_B} \frac{d^D k}{(2\pi/a)^D} e^{ikx} \tilde{f}(k)$$

where its Fourier transform is defined by

$$\tilde{f}(k) = \sum_x e^{-ikx} f(x)$$

The plane wave, e^{ipx} , viewed as an infinite dimensional vector indexed by the lattice sites x , is an eigenvector of the matrix $\Delta(x, y, \mu)$. To see this, one can easily check that it satisfies the eigenvalue equation

$$(36) \quad \sum_y [\Delta(x, y)] [e^{ipy}] = \left[\frac{J}{k_B T} \sum_i 4 \sin^2 p_i / 2 + \kappa^2 \right] [e^{ipx}]$$

It means that the eigenvalues of $\Delta(x, y)$ are $\left[\frac{J}{k_B T} \sum_i 4 \sin^2 p_i / 2 + \kappa^2 \right]$, one for each value of the wave-vector p in the Brillouin zone. The determinant of Δ is the product of its eigenvalues. It is most convenient to write this as the trace of the logarithm of the determinant which is readily then expressed as the sum over eigenvalues (which, for the lattice of infinite volume, N , is an integral with the appropriate normalization)

$$(37) \quad \begin{aligned} \text{Tr} \ln \frac{\Delta}{2\pi} &= N \int \frac{d^D p}{(2\pi/a)^D} \ln \frac{1}{2\pi} \left[\frac{J}{k_B T} \sum_i 4 \sin^2 p_i / 2 + \kappa^2 \right] \\ &= V \int \frac{d^D p}{(2\pi)^D} \ln \left[\frac{J}{k_B T} \sum_i 4 \sin^2 p_i / 2 + \kappa^2 \right] - N \ln 2\pi \end{aligned}$$

and the inverse of the matrix $\Delta(x, y)$ is gotten by finding that matrix whose eigenvalues are the inverses of the eigenvalues of $\Delta(x, y)$. That matrix is

$$(38) \quad \Delta^{-1}(x, y) = \int_{\Omega_B} \frac{d^D p}{(2\pi)^D} \frac{e^{ip(x-y)}}{\left[\frac{J}{k_B T} \sum_i 4 \sin^2 p_i / 2 + \kappa^2 \right]}$$