THE INFINITE RANGE ISING MODEL, LANDAU THEORY AND CRITICAL EXPONENTS

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1. Exact Solution of the Infinite Range Ising Model

The Infinite range Ising model is exactly solvable. It is a model where, an approximation to other versions of the Ising and other magnetic models, called "mean field theory" gives an exact solution of the model. In the infinite range model, every spin interacts equally with every other spin, regardless of their relative locations. The Hamiltonian is

$$H \ = \ -\frac{J}{2N}\sum_{x,y=1}^{N}\sigma_x\sigma_y - \sum_{x=1}^{N}B\sigma_x \tag{1}$$

The spin variables are σ_x with x ranging over all of the positions of the spins and as usual, each spin takes on one of two orientations, $\sigma_x = 1$ or $\sigma_x = -1$. We shall not have to be very specific about those positions or the dimension of the space or other configurational details since every spin interacts with every other spin with the same interaction strength. In fact, in the first term in the Hamiltonian, we can think of a spin σ_x as interaction with the average $\frac{1}{N}\sum_y \sigma_y$ of all of the spins. This means that this model will be similar and in fact we will see it is identical to an approximation to more sophistocated spin systems that is called mean field theory.

The variable B_x is the analog of an external magnetic field which couples to the magnetic moments of the system which are the spins. We will typically (but not always) specialize to the case of a constant external field, $B_n = B$. In this Hamiltonian each spin interacts with the sum of all of the rest of the spins. This the latter are large in number, the fluctuations of the expectation value of their sum is small and it resembles the interaction with a classical field, the "mean field" of the assembly of the remaining spins.

The partition function is given by the sum over all configurations of the spins, with each configuration weighted by the Boltzmann distribution.

$$Z[T,N,B] = \sum_{\rm spins} e^{-H/k_BT}$$

or, more explicitly,

$$Z[T,N,h] = \sum_{\rm spins}\exp\left(\frac{J}{2NK_BT}\sum_{xy}\sigma_x\sigma_y + \frac{B}{k_BT}\sum_x\sigma_x\right)$$

We can evaluate this sum. For this purpose, we begin with the integral

$$1=\sqrt{\frac{NJ}{2\pi k_BT}}\int_{-\infty}^{\infty}d\phi\exp\left(-\frac{NJ}{2k_BT}\phi^2\right)=\sqrt{\frac{NJ}{\pi k_BT}}\int_{-\infty}^{\infty}d\phi\exp\left(-\frac{NJ}{2k_BT}\left(\phi-\frac{1}{N}\sum_x\sigma_x\right)^2\right)$$

which we insert into the partition function so that it becomes

$$Z[T,B,N]=\sqrt{\frac{NJ}{\pi k_BT}}\int_{-\infty}^{\infty}d\phi\sum_{\text{spins}}e^{\frac{J}{2NK_BT}\sum_{xy}\sigma_x\sigma_y+\frac{B}{k_BT}\sum_x\sigma_x}e^{-\frac{NJ}{2k_BT}\left(\phi-\frac{1}{N}\sum_x\sigma_x\right)^2} \tag{2}$$

Upon combining the exponents, becomes

$$Z[T,B,N]=\sqrt{\frac{NJ}{2\pi k_BT}}\int_{-\infty}^{\infty}d\phi e^{-\frac{NJ}{2k_BT}\phi^2}\sum_{\text{spins}}e^{\frac{J\phi+B}{k_BT}\sum_x\sigma_x} \tag{3}$$

We can now do the sum over the spins explicitly

$$\sum_{\text{spins}}\prod_x e^{\frac{J\phi+B}{k_BT}\sigma_x}=\left(e^{\frac{J\phi+B}{k_BT}}+e^{-\frac{J\phi+B}{k_BT}}\right)^N \tag{4}$$

so that the partition function becomes

$$Z[T,B,N]=\sqrt{\frac{NJ}{2\pi k_BT}}\int_{-\infty}^{\infty}d\phi e^{-N\left[\frac{J}{2k_BT}\phi^2-\ln\left(2\cosh\left(\frac{J\phi+B}{k_BT}\right)\right)\right]} \tag{5}$$

We are interested in the limit when N is very large. In that case the saddle point technique can be used to approximate the integral up to power law accuracy in $1/N$. The essence of the approximation is simply to replace the integral by the value of the integrand evaluated at its maximum. To the leading order, the saddle point technique approximates the integral as

$$Z[T,B,N]\approx e^{-N\inf_\phi\left[\frac{J}{2k_BT}\phi^2-\ln\left(2\cosh\left(\frac{J\phi+B}{k_BT}\right)\right)\right]} \tag{6}$$

From this expression we obtain the Helmholtz free energy

$$F[T,B,N]=Nk_BT\inf_\phi\left[\frac{J}{2k_BT}\phi^2-\ln\left(\cosh\left(\frac{J\phi+B}{k_BT}\right)\right)\right]-Nk_BT\ln 2 \tag{7}$$

Corrections to this formula are by terms which are of order $\ln N$ and, relative to the contribution that we have kept, they can be ignored in the limit where $N\to\infty$. The function in the square brackets in the above equation goes to positive infinity when ϕ goes to positive or negative infinity and it is finite in between, so we expect that it has at least one minimum over the range of real values of ϕ. To find the infimum, we must first find the absolute minimum. An extremum which is a maximum or minimum must occur for a value of ϕ where the derivative of the function vanishes,

$$\text{first derivative}=\frac{J}{k_BT}\left[\phi-\tanh\left(\frac{J\phi+B}{k_BT}\right)\right]=0 \tag{8}$$

It is easy to confirm that this equation always has at least one solution. We do this by noting that, the function on the left-hand-side, $\phi - \tanh\left(\frac{J\phi+B}{k_BT}\right)$ goes to infinity as ϕ goes to infinity, minus infinity as ϕ toes to minus infinity and it is continuous in between. Therefore it should cross zero at least once.

The solution of (8) is a minimum if the second derivative

$$\begin{aligned} \text{second derivative} &= \frac{J}{k_BT} - \left(\frac{J}{k_BT}\right)^2 \frac{1}{\cosh^2\left(\frac{J\phi+B}{k_BT}\right)} \\ &= \frac{J}{k_BT} - \left(\frac{J}{k_BT}\right)^2 + \left(\frac{J}{k_BT}\right)^2 \tanh^2\left(\frac{J\phi+B}{k_BT}\right) \end{aligned} \tag{9}$$

is positive when it is evaluated on the solution. In fact, it is easy to see by inspection that the second derivative is always positive when the temperature is higher than a critical value $T > T_C = \frac{J}{k_B}$ or when B is of large enough magnitude. In those cases the function that we are seeking a minimum of is convex and it has a unique minimum.

On the other hand, when $T < T_C$, and when B is small enough, the curvature of the function has two regions where it is positive and one region where it is negative. This indicates that it has two minima and one maximum. In that case, we should choose one of the minima, the one for which the free energy is the lesser to be the physical solution. When $T \to T_C^-$ the three extrema become a degenerate triple root of the equation.

When $T \approx T_C$, inspection of the above equations tells us that the solutions for ϕ are small if B/k_BT is small. In that region, we can simplify the problem by Taylor expanding the terms in the free energy. Using

$$\ln\cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4$$

We find that

$$F[T \approx T_C, B, N] = Nk_BT \inf_\phi \left[\frac{J}{2k_BT}\phi^2 - \frac{1}{2}\left(\frac{J\phi+B}{k_BT}\right)^2 + \frac{1}{12}\left(\frac{J\phi+B}{k_BT}\right)^4 + \dots\right] - Nk_BT\ln 2 \tag{10}$$

We will study the behaviour of the free energy and some of the other thermodynamic attributes in the region of the phase diagram where $B = 0$ and $T \sim T_C$ and where $T = T_C$ and $B/k_BT \sim 0$. In the first of these, we obtain the free energy from

$$F[T \approx T_C, B = 0, N] = N \inf_\phi \left[J\left(1 - \frac{T_C}{T}\right)\frac{\phi^2}{2} + \frac{k_BT}{12}\left(\frac{T_C}{T}\right)^4 \phi^4 + \dots\right] - Nk_BT\ln 2 \tag{11}$$

The values of ϕ at the infimum are

$$\phi = \begin{cases} 0 & T > T_C \\ \pm\sqrt{\frac{3J}{k_BT}\left(\frac{T}{T_C}\right)^4\left(\frac{T_C}{T} - 1\right)} & T < T_C \end{cases} \tag{12}$$

First of all, we observe that ϕ is indeed small and our use the Taylor expansion of the free energy in equation (11) is justified a posteriori by this fact. It is effectively an expansion in $|1 - T/T_C|$ which we can tune to be arbitrarily small by tuning the temperature to be close to its critical value. Then, (13) could be written as

$$\phi = \begin{cases} 0 & T > T_C \\ \pm\sqrt{3\left(\frac{T_C}{T} - 1\right)} + \ldots & T < T_C \end{cases} \tag{13}$$

where the ellipses represent corrections which go to zero as $T \to T_C$ faster than the terms that are already displayed.

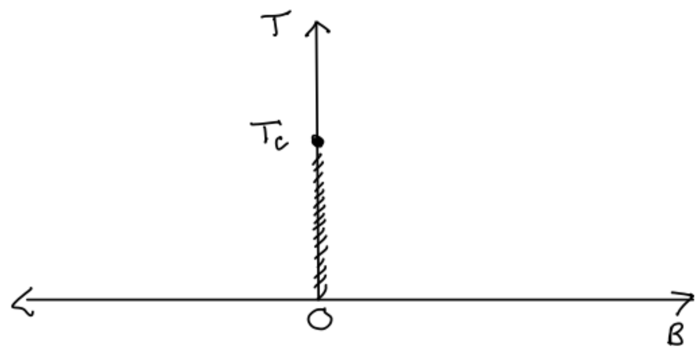

FIGURE 1. The phase diagram of the infinite range Ising model is depicted. There is a line of first order phase transitions located on the segment between zero and T_C of the $B = 0$ axis, denoted by the crosshatch. The line ends at a second order phase transition at the critical point $(T, B) = (T_C, 0)$.

Using this solution, we find the the Helmholtz free energy

$$F[T \approx T_C, B = 0, N] = \begin{cases} -Nk_BT\ln 2 & T > T_C \\ -Nk_BT\ln 2 - \frac{3J}{4}\left(\frac{T}{T_C} - 1\right)^2 + \ldots & T < T_C \end{cases} \tag{14}$$

We see that the free energy is continuous function of T and it has a continuous first derivative by T. However, the second derivative by T exists everywhere but it is discontinuous at T_C. We then say that it exhibits a "second order phase transition" at the critical point $(T, B) = (T_C, 0)$. This critical point is depicted in the phase diagram is figure 1.

The singularity of the Helmholtz free energy is reflected in the specific heat which is defined by

$$c = -\frac{T}{N}\frac{\partial^2}{\partial T^2}F[T \approx T_C, B = 0, N] = \begin{cases} 0 + \ldots & T > T_C \\ \frac{3J^2}{2k_BT_C} + \ldots & T < T_C \end{cases} \tag{15}$$

where it can have a singularity and it generally scales like a power law as $T \to T_C$,

$$c \sim \begin{cases} (T/T_C - 1)^{-\alpha} & T > T_C \\ (1 - T/T_C)^{-\alpha'} & T < T_C \end{cases} \tag{16}$$

From equation (15) we find that the critical exponents for both limits, from above and below the transition, are zero,

$$\alpha = 0,\ \alpha' = 0 \tag{17}$$

The magnetization is obtained by taking a derivative of the Helmholtz free energy by the magnetic field, B,

$$M = -\left.\frac{\partial F}{\partial B}\right|_{T,N} \tag{18}$$

Using this formula, we see that the magnetization density is given by

$$m = \tanh\left(\frac{J\phi + B}{k_B T}\right) = \phi \tag{19}$$

In taking the derivative needed to obtain the above equation we noted that the derivative by ϕ must vanish when it solves the equation for the minimum and we have used the equation for ϕ to conclude that ϕ, the solution of equation (8) is equal to the magnetization density. Then, equation (13) tells us that, as the temperature approaches the critical temperature from below, $T \to T_C^-$, the magnetization scales to zero like a power law with a critical exponent

$$m \sim (1 - T/T_C)^\beta \tag{20}$$

and we have found the value of that critical exponent,

$$\beta = \frac{1}{2} \tag{21}$$

The magnetic susceptibility is defined as the derivative of the magnetization density by the magnetic field,

$$\chi = \left.\frac{\partial m}{\partial B}\right|_{T,N} \tag{22}$$

We can find it by taking a derivative of the equation which determines ϕ, equation (8), by B to get

$$\frac{\partial \phi}{\partial B} - \left(\frac{J}{k_B T}\frac{\partial \phi}{\partial B} + \frac{1}{k_B T}\right)\left(1 - \tanh^2\left(\frac{J\phi + B}{k_B T}\right)\right) = 0$$

$$\chi = \frac{\frac{1}{k_B T}(1 - m^2)}{1 - \frac{T_C}{T}(1 - m^2)} \tag{23}$$

The magnetic susceptibility generally becomes singular as the phase transition is approached,

$$\chi \sim \begin{cases} (T/T_C - 1)^{-\gamma} & T > T_C \\ (1 - T/T_C)^{-\gamma'} & T < T_C \end{cases} \tag{24}$$

If we study our solution for the susceptibility given in equation (23) we find the Curie-Weiss law

$$\chi \sim \begin{cases} \frac{1}{(T/T_C-1)} & T\to T_C^+ \\ \frac{1}{(1-T/T_C)} & T\to T_C^- \end{cases} \tag{25}$$

and the critical exponents are

$$\gamma = 1 \ , \ \gamma' = 1 \tag{26}$$

Finally, as well as setting $B = 0$ and approaching the phase transition by varying the temperature near T_C, we can set $T = T_C$ and vary B near zero. In that regime, the magnetization density and the magnetic field are generally related by

$$B \sim m^\delta \tag{27}$$

which defines a critical exponent δ. If we examine the equation for ϕ and we set $T = T_C$ there, we see that ϕ is non-sero due to the presence of B and also that

$$m \sim |B|^{\frac{1}{4}}\text{sign}(B) \tag{28}$$

which tells us that the critical exponent is

$$\delta = 3 \tag{29}$$

We can also see that at lower temperatures, $T < T_C$, the magnetization is nonzero when B is nonzero and it has the same sign as B. However, as B is decreased to zero the magnetization does not go to zero, but goes to a nonzero value, that is, there is spontaneous magnetization and spontaneously broken Z_2 symmetry at that point. The magnetization there inherits its sign from the sign of values of B through which the line is approaches. Then when B changed further and crosses to values of the opposite sign, the magnetization flips sign. This discontinuity of the magnetization is a discontinuity of the first derivative of the free energy by B and it therefore corresponds to a first order phase transition. We thus conclude that there is a line of first order phase transitions on the interval $0 \leq T < T_C$ of the $B = 0$ axis. This is indicated on the phase diagram in figure 1.

2. Critical exponents

The various scaling behaviours that we have observed to be associated with a second order phase transition are universal in that they hold true for any phase transition that is governed by mean field theory. This includes the infinite range Ising model which we have studied here and it includes Landau theory which we will study shortly.

The critical exponents are defined by the following table

(30)

temperature	$\tau \equiv T - T_C$	$\tau < 0$	$\tau > 0$
specific heat	$c = -\frac{T}{N}\frac{\partial^2 F}{\partial T^2}$	$c \sim (-\tau)^{-\alpha'}$	$c \sim (\tau)^{-\alpha}$
order parameter	m	$m \sim (-\tau)^{-\beta'}$	$m = 0$
susceptibility	$\chi = \frac{\partial m}{\partial B}$	$\chi \sim (-\tau)^{-\gamma'}$	$\chi \sim (\tau)^{-\gamma}$
correlation length	$< m(x)m(0) >\sim e^{-\lvert x\rvert/\xi}$	$\xi \sim (-\tau)^{-\nu'}$	$\xi \sim (\tau)^{-\nu}$

Then, when $T = T_C$ and $B \sim 0$, there is an exponent defined by

$$T = T_C : \ B \sim m^\delta \tag{31}$$

Finally, the correlation function is long ranged with a power law falloff which is characterized by a critical exponent

$$T = T_C : \ g(x) \sim \frac{1}{|x|^{d-2+\eta}} \tag{32}$$

The results for the infinite range Ising model and for mean field theory are

$$\alpha = 0, \ \alpha' = 0, \ \beta = 1, \ \gamma = 1, \ \gamma' = 1, \ \delta = 3 \tag{33}$$

Some extrapolation beyond strict mean field theory, called Landau theory, also determines

$$\eta = 0 \ , \ \nu = \nu' = \frac{1}{2}$$

3. Landau Theory

We might note that very little of the details of the Ising model really mattered when we found the critical exponents. All that really mattered were the first few terms in a Taylor expansion fo the function that was to be minimized to find the magnetization and determine the free energy. And even the magnitude of the coefficient of the ϕ^4 term had to be positive but otherwise its value didn't change the nature of the phase transition. We can abstract these essential fieatures to formulate a phenomenological theory of second order phase transitions called Landau theory, which is originally due to L.D.Landau.

Landau theory states that we should determine the essential contributions to the free energy by finding the absolute minumum of the quantity

$$F = \inf_\phi \int d^D x \left\{ \frac{1}{2} \vec{\nabla}\phi \cdot \vec{\nabla}\phi + \frac{\tau}{2}\phi^2(x) + \frac{\lambda}{4!}\phi^4(x) - B(x)\phi(x) \right\} \tag{34}$$

where our only assumptions are that $\tau \sim T - T_C$ depends linearly on the temperature as the temperature approaches the critical temperature and that the second constant $\lambda > 0$. We have made allowance for the possibility that the magnetization and the magnetic filed can have a mild dependence on position in space and we thus allow $\phi(x)$ to depend on the position. We assume this position dependence is so smooth that the substructure, for example the underlying lattice, is indiscernible and that $\phi(x)$ is a smoothly varying function. Aside from the terms that were in the energy functional for the infinite range Ising model, $\frac{\tau}{2}\phi^2(x) + \frac{\lambda}{4!}\phi^4(x) - B(x)\phi(x)$ we have added a term with derivatives $\frac{1}{2}\vec{\nabla}\phi(x) \cdot \vec{\nabla}\phi(x)$ which favours smooth field configurations. The integral over the spatial volume, $\int dx$ takes the place of the factor N. Then, to find the τ-dependence of the free energy we need to find the function $\phi(x)$ for which the energy functional in equation (34) is minimal.

Finding the $\phi(x)$ which minimizes the functional in (34) is straightforward. Like in the infinite range Ising model, we are interested in the behaviour of the system when $N = 0$

and $T \sim T_C$ or else when $T = T_C$ and $B \sim 0$. Let us first assume that $B = 0$ and we want to find the minimum of

$$\int d^D x \left\{ \frac{1}{2} \vec{\nabla}\phi \cdot \vec{\nabla}\phi + \frac{\tau}{2}\phi^2(x) + \frac{\lambda}{4!}\phi^4(x) \right\} \tag{35}$$

If $\tau > 0$ we note that every term in the integrand is positive so an absolute minimum occurs when $\phi(x) = 0$. If $\tau < 0$ we rewrite the integral as

$$\int d^D x \left\{ \frac{1}{2} \vec{\nabla}\phi \cdot \vec{\nabla}\phi + \frac{\tau}{2}\phi^2(x) + \frac{\lambda}{4!} \left(\phi^2(x) + \frac{4!}{4\lambda}\tau \right)^2 - \frac{4!}{16\lambda}\tau^2 \right\} \tag{36}$$

What results is a sum of three positive terms and a negative constant ϕ-independent term. Ahand we find the absolute minimum by setting the three positive terms to zero. This implies that ϕ is a constant, independent of x and

$$\phi = \pm\sqrt{-\frac{4!}{4\lambda}\tau} \tag{37}$$

The free energy is then

$$F = \begin{cases} 0 & \tau > 0 \\ -V\frac{4!}{16\lambda}\tau^2 & \tau < 0 \end{cases} \tag{38}$$

which is identical to what we found for the infinite ranged Ising model. In fact all of the critical behaviour that we found there are easily reproduced.

Beyond the mean field theory description of the second order phase transition that we found for the infinite ranged Ising model, the Landau theory is postulated to contain information about correlation functions which are defined as [1]

$$m(x) = -\frac{\delta F}{\delta B(x)} \tag{40}$$

and the connected correlation function is

$$< m(x)m(y) > - < m(x) >< m(y) >= \frac{\delta^2 F}{\delta B(x)\delta B(y)} = \frac{\delta m(x)}{\delta B(y)} \tag{41}$$

Even when we are studying the correlation function with $B = 0$, the idea is that we find the second functional derivative by B as above and put B to zero afterwards.

Applying the functional derivative to the free energy functional, we obtain

$$m(x) = \phi(x) \tag{42}$$

The magnetization density is equal to the function $\phi(x)$ which minimizes the free energy.

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[1] To take a functional derivative by a function $\phi(x)$ of a quantity $F[\phi]$ containing the function ϕ, $\frac{\delta F}{\delta\phi(x)}$, we linear term in ϵ is the functional derivative. For example, using this prescription

$$\frac{\delta}{\delta\phi(x)} \int d^D y \phi^n(y) = n\phi^{n-1}(x),\ \frac{\delta}{\delta\phi(x)} \int d^D y \vec{\nabla}\phi(y) \cdot \vec{\nabla}\phi(y) = -\vec{\nabla}^2\phi(x),\ \frac{\delta}{\delta\phi(x)}\phi(y) = \delta(x-y) \tag{39}$$

By a similar procedure we can also cast our minimum problem by taking a functional derivative of the free energy by ϕ and then setting that derivative to zero. The equation which determines ϕ is thus

$$\left(-\vec{\nabla}^2+\tau+\frac{\lambda}{3!}\phi^2(x)\right)\phi(x)=B(x) \tag{43}$$

If we take a further functional derivative of the above equation by B we fund an equation for the correlation function

$$\left(-\vec{\nabla}^2+\tau+\frac{\lambda}{2}\phi^2(x)\right)\frac{\phi(x)}{\delta B(y)}=\delta(x-y) \tag{44}$$

and the connected correlation function is

$$< m(x)m(y) > - < m(x) >< m(y) >= (x|\frac{1}{-\vec{\nabla}^2+\tau+\frac{\lambda}{2}\phi^2(x)}|y) \tag{45}$$

where the expression on the right-hand-side is a formal expression for the inverse of the differential operator, that is, the Green function. In the phase where $B=0,\tau>0,\phi=0$ we can use the Fourier transform to get

$$< m(x)m(y) > - < m(x) >< m(y) >= \int\frac{d^Dk}{(2\pi)^D}\frac{i^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2+\tau}\sim e^{-\sqrt{\tau}|\vec{x}-\vec{y}|} \tag{46}$$

and the correlation length is

$$\xi=1/\sqrt{\tau} \tag{47}$$

When $B=0,\tau<0$ and $\phi^2=-6\tau/\lambda$,

$$< m(x)m(y) > - < m(x) >< m(y) >= \int\frac{d^Dk}{(2\pi)^D}\frac{i^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2-2\tau}\sim e^{-\sqrt{-2\tau}|\vec{x}-\vec{y}|} \tag{48}$$

and the correlation length is

$$\xi=1/\sqrt{-2\tau} \tag{49}$$

The critical exponents are

$$\nu=\nu'=\frac{1}{2} \tag{50}$$

These are the mean field theory values of the critical exponents for the correlation length.

Then, when $\tau=0$,

$$< m(x)m(y) > - < m(x) >< m(y) >= \int\frac{d^Dk}{(2\pi)^D}\frac{i^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2}\sim\frac{1}{|\vec{x}-\vec{y}|^{D-2}} \tag{51}$$

and the critical exponent $\eta=0$.

In the above discussion we have formulated the Landau theory to describe the Ising model. The main features of the model were

(1) The dimension of the space on which the model is defined.

(2) The symmetry of the model. In this case it is the Z_2 symmetry of the Ising model when the magnetic field is put to zero. The Z_2 symmetry transforms $\phi \to -\phi$.
(3) It contains the term $\tau\phi^2/2$ and $\tau = 0$ at the critical temperature. Also, τ varies as a linear function of the temperature as it approaches zero.
(4) The parameter λ in the ϕ^4 term is positive.

If we were to apply Landau theory to a magnetic system other than the Ising model, the main change would be to replace the variable ϕ by a variable which transforms in the appropriate way under the symmetries of the model. For example, if in the spin model, the spins, rather than simply being 1 or -1, could take any orientation so that it would be described by a unit vector $\hat{\sigma}_x$ which has three components and has $\vec{\sigma}_x^2 = 1$, and if the Hamiltonian with $B = 0$ has symmetry under arbitrary similtaneous rotations of all of the spins, the appropriate Landau theory would have the free energy

$$F = \int d^D x \left\{ \frac{1}{2} \sum_{a=1}^{3} \vec{\nabla}\phi^a \cdot \vec{\nabla}\phi^a + \frac{\tau}{2}\vec{\phi}^2 + \frac{\lambda}{4}(\vec{\phi}^2)^2 - \vec{B} \cdot \phi \right\} \tag{52}$$

Aside from a different symmetry breaking scheme, the critical exponents for this model are still identical to those of mean field theory that we listed in the previous section.