Tutorial 2: The $O(3)$ Vector Model

1 Introduction to the O(3) vector model

The $O(3)$ vector model is a simple model of a magnetic system which has spin variables $\hat{n}(x)$ which are three-component unit vectors . They obey

$$\hat{n}(x) \cdot \hat{n}(x) = 1 \ , \ \forall x$$

For an individual spin, we can always use spherical polar coordinates to describe the spin unit vector

$$\hat{n}(x) = (\sin\theta(x)\cos\phi(x), \sin\theta(x)\sin\phi(x), \cos\theta(x)) \ , \ 0 \le \phi(x) < 2\pi, \ 0 \le \theta(x) \le \pi$$

We will generally assume that the spins reside on a cubic lattice in D dimensions. The sites of such a lattice are labeled by vectors in D dimensions which have discrete values

$$x = a\left(k_1\hat{1} + k_2\hat{2} + \ldots + k_D\hat{D}\right)$$

where $\hat{i}$ for $i = 1, ..., D$ are unit vectors aligned with the D cartesian coordinate axes in D-dimensional euclidean space and $(k_1, k_2, ..., k_D)$ are integers.

The energy of a configuration of the spins is given by the Hamiltonian

$$H = -J\sum_x\sum_{i=1}^{D}\hat{n}(x)\cdot\hat{n}(x + a\hat{i}) - \sum_x \vec{B}\cdot\hat{n}(x) \quad (1)$$

Like the Ising model which we have already studied, the first term in the Hamiltonian favours the alignment of neighbouring spins. The sum over the positions in that term counts each set of nearest neighbours once. The second term in the Hamiltonian favours alignment of the spins with an external magnetic field $\vec{B}$ which is itself now a three-vector, $\vec{B} = (B_1, B_2, B_3)$. We will assume that the total number of spins is a very large integer, N, and that the volume of the lattice is therefore given by $V = Na^D$.

The main difference with the Ising model is the possible orientations of the spins. For the Ising model there are only two orientations of the spin. For the O(3) model there are an infinite number. When the external field is switched off, the first term in the Hamiltonian of the Ising model has Z_2 symmetry whereas the first term in the Hamiltonian of the O(3) model has O(3) symmetry. The transformations of O(3) symmetry include all possible rotations of the vector $\hat{n}$ about the origin. It also contains a reflection $\hat{n} \to -\hat{n}$. The first tern in the Hamilton contains only scalar products, $\hat{n}(x) \cdot \hat{n}(x + a\hat{i})$ and it is therefore symmetric under simultaneous rotations or reflections of all of the spins at the same time. The set of all such rotations comprises the Lie group SO(3). When reflections as well as rotations are included, the group is called O(3). This is the origin of the name of the model.

The magnetic coupling in the second term in the Hamiltonian is not symmetric under O(3). The presence of the magnetic field $\vec{B}$ biases the system so that the spins tend to align with the vector $\vec{B}$.

Let us begin by studying the lowest energy state of this model in the special case where $\vec{B} = 0$. The lowest energy state of the O(3) model in that case is the state which minimizes the first term in the Hamiltonian. We will sometimes use the quantum mechanical term "ground state" for the lowest energy state (even though we are definitely not studying quantum mechanics here, everything is classical). The ground state has all of the spins aligned. This configuration therefore has $\hat{n}(x) = \hat{n}$ where $\hat{n}$ is an x-independent unit vector. However the direction of this joint alignment of the spins, $\hat{n}$, is still arbitrary. Unlike the Ising model, where there were only two possibilities for the ground state, with all of the spins aligned as $\sigma_x = 1$ or $\sigma_x = -1$, in this case there are infinite possibilities given by the possible spin polarizations. The possibilities are infinite because the unit vector $\hat{n}$ can have any orientation. With a fixed orientation this state emulates the magnetized state of an isotropic Ferromagnet with spontaneous magnetization

$$\vec{M} = N\hat{n}$$

1. *Show that, when $\vec{B} = 0$, the energy of the ground states are all equal to*

$$E_0 = -NDJ$$

Unlike the case of the Ising model, there is no gap between the ground state and the states with higher energy. To obtain a higher energy state we only need to misalign one of the spins slightly. That increases the energy and both the misalignment and the energy that is the result of the misalignment can be infinitesimally small.

In analogy with the way in which a ground state of the Ising model (with $B = 0$) breaks its Z_2 symmetry and we call it a phase with spontaneous symmetry breaking, a ground state of the O(3) model with $\vec{B} = 0$ breaks the O(3) symmetry. There is a residual symmetry group. When we choose the ground state with $\hat{n}(x) = \hat{n}$, it is still symmetric under rotations where the rotation axis is $\hat{n}$ itself, that is, the set of all rotations which leave $\hat{n}$ unchanged. There are also reflections through the plane which contains $\hat{n}$. These transformations comprise the Lie group O(2) which is a residual symmetry of the ground state. We say that O(3) symmetry is broken to O(2). The set of possible ground states is the set of possible orientations of $\hat{n}$ which is in one-to-one correspondence with the set of points on the unit 2-sphere S^2 and it is sometimes also written as the factor group

$$S^2 = O(3)/O(2) \tag{2}$$

If we turn on a nonzero value for $\vec{B}$, it biases the ground states so that the lowest energy one is now the one with

$$\hat{n} = \frac{\vec{B}}{|\vec{B}|} \tag{3}$$

This is now a unique ground state. The Helmholtz free energy must coincide with the ground state energy when the temperature is zero.

2. *Show that the Helmholtz free energy evaluated at zero temperature, is given by*

$$F(T = 0, \vec{B}, N) = -NDJ - N|\vec{B}| \tag{4}$$

This free energy, as a function of $\vec{B}$ and N is not analytic at $\vec{B} = 0$. It is differentiable there. Remember that the magnetization is defined as

$$M^a = -\frac{\partial}{\partial B^a} F(T, \vec{B}, N)\Big|_{T,N} \tag{5}$$

3. *Show that the derivative (gradient) by of the expression for the free energy in equation (4) by the vector $\vec{B}$ in equation (5) recovers the magnetization*

$$\lim_{T\to 0}\vec{M} = N\frac{\vec{B}}{|\vec{B}|} = N\hat{n} \tag{6}$$

However the second derivatives of F by the components of $\vec{B}$ are singular at $\vec{B}=0$. Since only the first derivative exists, $(T,\vec{B})=(0,0)$ is a critical point where there is a first order phase transition. It is also at the point where we find the spontaneous breaking of the O(3) symmetry with spontaneous magnetization with direction depending on the direction in the space of $\vec{B}$'s that we approach $\vec{B}=0$ from.

Now, let us turn on the temperature. The partition function of the O(3) vector model is

$$Z[T,B,N] = \int [dn(x)]\ \exp\left\{-\frac{1}{k_BT}H[\phi]\right\} \tag{7}$$

$$= \int [dn(x)]\ \exp\left\{\frac{J}{k_BT}\sum_{x,i} n(x)\cdot n(x+\hat{\imath}) + \frac{1}{k_BT}\cdot\sum_x B\cdot n(x)\right\} \tag{8}$$

where the integration symbol $\int [dn(x)]$ is a shorthand for the very large number of integrals

$$\int [dn(x)]\ldots \equiv \prod_x \left[\int_0^{2\pi} d\phi(x)\int_0^{\pi}\sin\theta(x)d\theta(x)\right]\ldots \tag{9}$$

where we have one integral over the unit 2-sphere, written here in polar coordinates, for each of the N spins in the system. This is how we average over all spin configurations of the O(3) model.

2 The one-dimensional O(3) model

Let us consider the simple example of the O(3) spin chain in one dimension. The Hamiltonian is

$$H = -J \sum_{x=1}^{N-1} \hat{n}(x) \cdot \hat{n}(x+1) \tag{10}$$

where we assume we are using units where the lattice constant is equal to one. We will use open boundary conditions where we integrate over the states at the last sites at each end of the spin chain. We have set the external field to zero, $\vec{B} = 0$. It turns out that this model is easily solvable only in that case and we will focus on it here. The partition function is the sum (integral) over spin states of the Boltzmann weight,

$$\begin{aligned} Z[T, \vec{B} = 0, N] = \\ \int_0^{2\pi} d\phi(1) \int_0^{\pi} \sin\theta(1) d\theta(1) \ldots \int_0^{2\pi} d\phi(N) \int_0^{\pi} \sin\theta(N) d\theta(N) \times \\ \times e^{\frac{J}{k_B T}\hat{n}(1)\cdot\hat{n}(2)} e^{\frac{J}{k_B T}\hat{n}(2)\cdot\hat{n}(3)} \ldots e^{\frac{J}{k_B T}\hat{n}(N-1)\cdot\hat{n}(N)} \end{aligned} \tag{11}$$

Now, we could do the integral over the first spin, $\hat{n}(1)$.

4. *Show that the result of doing the integral over the first spin,* $\hat{n}(1)$ *is*

$$\int_0^{2\pi} d\phi(1) \int_0^{\pi} \sin\theta(1) d\theta(1) e^{\frac{J}{k_B T}\hat{n}(1)\cdot\hat{n}(2)} = 4\pi \frac{k_B T}{J} \sinh \frac{J}{k_B T} \tag{12}$$

which is independent of $\hat{n}(2)$. Using this result, we can then do the integral over $\hat{n}(2)$ to get a similar factor and continue to do the integrals over all of the spins. The result is

$$Z[T, \vec{B} = 0, N] = 4\pi \left[4\pi \frac{k_B T}{J} \sinh \frac{J}{k_B T} \right]^{N-1} \tag{13}$$

and the Helmholtz free energy is

$$F[T, \vec{B} = 0, N] = -(N-1) k_B T \ln \left[\frac{k_B T}{J} \sinh \frac{J}{k_B T} \right] - N k_B T \ln(4\pi) \tag{14}$$

Note that we have not made any approximation yet. In particular, N has not been taken to be large and that is why we have kept the factor $N-1$ in front of the free energy. We can solve this simple model for any value of N.

Like the one-dimensional Ising model, the one-dimensional O(3) model is analytic in the regime $\vec{B}=0$ and $T>0$. There cannot be a phase transition there. It does, however, have a complicated zero temperature limit.

5. *Show that an asymptotic expansion of the Helmholtz free energy about zero temperature is*

$$F[T,\vec{B}=0,N] \sim -(N-1)J-(N-1)k_BT\ln\frac{k_BT}{2J}-Nk_BT\ln(4\pi)+\ldots$$

What is the first (most important) correction to this formula?

The zero temperature limit of the free energy given in the expression above is singular in that its first derivative diverges logarithmically at $T=0$. This is consistent with the existence of a phase transition at $T=0,\vec{B}=0$.

We could also compute a correlation function of the spin variables. For this we need to consider

$$\begin{aligned}
&< n^a(x)n^b(y) >= \frac{1}{Z[T,\vec{B}=0,N]}\times \\
&\times\int_0^{2\pi}d\phi(1)\int_0^{\pi}\sin\theta(1)d\theta(1)\ldots\int_0^{2\pi}d\phi(N)\int_0^{\pi}\sin\theta(N)d\theta(N)\times \\
&\times e^{\frac{J}{k_BT}\hat{n}(1)\cdot\hat{n}(2)}e^{\frac{J}{k_BT}\hat{n}(2)\cdot\hat{n}(3)}\ldots n^a(x)e^{\frac{J}{k_BT}\hat{n}(x)\cdot\hat{n}(x+1)}\ldots n^b(y)e^{\frac{J}{k_BT}\hat{n}(y)\cdot\hat{n}(y+1)}\ldots e^{\frac{J}{k_BT}\hat{n}(N-1)\cdot\hat{n}(N)}
\end{aligned} \tag{15}$$

We can begin to do this integral in the same way that we did to find the partition function. We would first integrate over $\hat{n}(1)$, then $\hat{n}(2)$ and so on, each time producing a factor of $4\pi\frac{k_BT}{J}\sinh\frac{J}{k_BT}$. We continue until we encounter the integral

$$\int_0^{2\pi}d\phi(x)\int_0^{\pi}\sin\theta(x)d\theta(x)n^a(x)e^{\frac{J}{k_BT}\hat{n}(x)\cdot\hat{n}(x+1)}$$

This is a straightforward integral to do analytically.

6. *Show that the result of doing the integral in the equation above is*

$$\int_0^{2\pi} d\phi(x) \int_0^{\pi} \sin\theta(x) d\theta(x) n^a(x) e^{\frac{J}{k_BT}\hat{n}(x)\cdot\hat{n}(x+1)} =$$
$$= `4\pi\left(\frac{k_BT}{J}\cosh\frac{J}{k_BT} - \left(\frac{k_BT}{J}\right)^2 \sinh\frac{J}{k_BT}\right) n^a(x+1) \qquad (16)$$

We then have a similar integral over $\hat{n}(x+1)$ and subsequently a similar integral over $\hat{n}(x+2)$ until we arrive at y where the integration will be

$$\int_0^{2\pi} d\phi(y) \int_0^{\pi} \sin\theta(y) d\theta(y) n^a(y) n^b(y) e^{\frac{J}{k_BT}\hat{n}(y)\cdot\hat{n}(y+1)}$$

Then we begin integrating at the other end of the spin chain by integrating over $\hat{n}(N)$ and $\hat{n}(N-1)$ etc until we arrive at the integral over $\hat{n}(y+1)$. That integral is independent of $\hat{n}(y)$ and the above integral then takes the form

$$\int_0^{2\pi} d\phi(y) \int_0^{\pi} \sin\theta(y) d\theta(y) n^a(y) n^b(y) = \frac{4\pi}{3}\delta^{ab}$$

7. *Show that the result of the computation that we have outlined above is*

$$< n^a(x) n^b(y) >=$$
$$\frac{1}{3}\delta^{ab}\frac{4\pi\left[4\pi\frac{k_BT}{J}\sinh\frac{J}{k_BT}\right]^{N+x-y}\left[4\pi\left(\frac{k_BT}{J}\cosh\frac{J}{k_BT} - \left(\frac{k_BT}{J}\right)^2\sinh\frac{J}{k_BT}\right)\right]^{y-x-1}}{4\pi\left[4\pi\frac{k_BT}{J}\sinh\frac{J}{k_BT}\right]^{N-1}}$$
$$= \frac{1}{3}\delta^{ab}\left(\coth\frac{J}{k_BT} - \frac{k_BT}{J}\right)^{|x-y|} \qquad (17)$$

The correlation length is usually deduced from the connected correlation function that is

$$< n^a(x) n^b(y) >_C = < n^a(x) n^b(y) > - < n^a(x) >< n^b(x) > \qquad (18)$$

To finish our computation we need to compute $< n^a(x) >$.

8. *Compute* $< n^a(x) >$ *and then show that the correlation length is given by*

$$\xi = -\frac{1}{\ln\left(\coth\frac{J}{k_BT} - \frac{k_BT}{J}\right)} \qquad (19)$$

We note that the correlation length is finite and non-zero at any positive value of the temperature. This supports our previous conclusion, drawn from analyticity of the free energy, that nothing dramatic happens in the regime $\vec{B} = 0, T > 0$. As the temperature is relaxed to zero, ξ diverges,

$$\lim_{T \to 0} \xi \sim \frac{J}{k_B T}$$

which is consistent with existence of a phase transition at zero temperature, at the critical point $\vec{B} = 0, T = 0$. Note that the divergence is power law and the correlation length has a well defined critical exponent there, defined by

$$\lim_{T \to T_C} \xi(T) \sim (T - T_C)^{-\nu}$$

and we have computed the critical exponent

$$\nu = -1 \tag{20}$$

3 The infinite range O(3) model

The infinite range O(3) model is also exactly solvable. Like the infinite range Ising model, it will turn out to have the same critical behaviour as mean field theory. The Hamiltonian is given by

$$H = -\frac{J}{2N} \sum_{x,y}^{N} \hat{n}(x) \cdot \hat{n}(y) - \sum_{x} \vec{B} \cdot \hat{n}(x) \tag{21}$$

In this model, every spin interacts with every other spin with the same interaction energy. The first term in the Hamiltonian favours alignment of the spins and the second term favours spins which are aligned with the magnetic field $\vec{B}$. The partition function is

$$Z[T, B, N] = \int [dn(x)] e^{\frac{J}{2N k_B T} \sum_{x,y} \hat{n}(x) \cdot \hat{n}(y) + \frac{1}{k_B T} \sum_x \vec{B} \cdot \hat{n}(x)} \tag{22}$$

To solve this model, we note that

$$e^{\frac{J}{2Nk_BT}\sum_{x,y}\hat{n}(x)\cdot\hat{n}(y)+\frac{1}{k_BT}\sum_x \vec{B}\cdot\hat{n}(x)} = e^{\frac{NJ}{2k_BT}\left(\frac{1}{N}\sum_x \hat{n}(x)\right)^2+\frac{1}{k_BT}\sum_x \vec{B}\cdot\hat{n}(x)}$$

We use the identity

$$1 = \left(\frac{J}{2\pi N k_B T}\right)^{\frac{3}{2}} \int [d\vec{\chi}] e^{-\frac{NJ}{2k_BT}\left(\vec{\chi}-\frac{1}{N}\sum_x \hat{n}(x)\right)^2} \tag{23}$$

where the integration $\int[d\vec{\chi}]...$ is defined similarly to $\int[dn]...$ and can be taken to be the integral in polar coordinates $\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta...$ where $\vec{\chi} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

9. *Plug the identity (23) into the partition function and show that the result can be written as the expression*

$$Z[T,B,N] = \left(\frac{J}{2\pi N k_B T}\right)^{\frac{3}{2}} \int d\vec{\chi} e^{-\frac{NJ}{2k_BT}\vec{\chi}^2} \prod_x \left[\int [dn(x)] e^{\frac{\vec{B}+J\vec{\chi}}{k_BT}\cdot\hat{n}(x)}\right] \tag{24}$$

Now, the integrations over the spins can be done explicitly.

10. *Show that, upon doing the integrations over the spins, we get*

$$\begin{aligned} Z[T,B,N] &= \left(\frac{J}{2\pi N k_B T}\right)^{\frac{3}{2}} \int d\vec{\chi} e^{-\frac{NJ}{2k_BT}(\vec{\chi})^2} \left[4\pi \frac{\sinh\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}{\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}\right]^N \\ &= \left(\frac{J}{2\pi N k_B T}\right)^{\frac{3}{2}} \int d\vec{\chi} e^{-N\left\{\frac{J}{2k_BT}\vec{\chi}^2 - \ln\left[4\pi\frac{\sinh\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}{\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}\right]\right\}} \end{aligned} \tag{25}$$

Now, we shall consider the limit where N is very large and evaluate the integral using the saddle point technique. The result is the expression for the free energy

$$F[T,B,N] = k_B T N \inf_M \left\{\frac{J}{2k_BT}\vec{\chi}^2 - \ln\left[\frac{\sinh\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}{\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}\right]\right\} + k_B T N \ln(4\pi) \tag{26}$$

where we have dropped contributions that grow at rates slower than N as N gets large.

11. *Why would you expect that the function*

$$f(\vec{\chi}) = \frac{J}{2k_BT}\vec{\chi}^2 - \ln\left[\frac{\sinh\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}{\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}\right] \tag{27}$$

has a minimum for at least one value of $\vec{\chi}$ *for any* $T > 0$*?*

12. *Show that the magnetization density is given by*

$$\vec{m} = \frac{\vec{B}+J\vec{\chi}}{|\vec{B}+J\vec{\chi}|}\left[\coth\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right| - \frac{1}{\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}\right] \tag{28}$$

evaluated on the value of $\vec{\chi}$ *where the function* $f(\vec{\chi})$ *in equation (27) is at an absolute minimum.*

The value of $\vec{\chi}$ where the function $f(\vec{\chi})$ in equation (27) is at an absolute minimum should satisfy the equation which is obtained by setting the derivative of that function by the components of χ to zero,

13. *Show that* χ *must satisfy the equation*

$$\vec{\chi} = \frac{\vec{B}+J\vec{\chi}}{|\vec{B}+J\vec{\chi}|}\left[\coth\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right| - \frac{1}{\left|\frac{\vec{B}+J\vec{\chi}}{k_BT}\right|}\right] \tag{29}$$

Upon comparing this equation with equation (30) we see that the magnetization density $\vec{m}$ is equal to the value that $\vec{\chi}$ takes at the infimum of $f(\vec{\chi})$. Then, we know that the magnetization is equal to the value of $\vec{\chi}$ at the minimum, so that we could also say that the magnetization obeys the equation

$$\vec{m} = \frac{\vec{B}+J\vec{m}}{|\vec{B}+J\vec{m}|}\left[\coth\left|\frac{\vec{B}+J\vec{m}}{k_BT}\right| - \frac{1}{\left|\frac{\vec{B}+J\vec{m}}{k_BT}\right|}\right] \tag{30}$$

Characterization of the.solutions of the equation for $\vec{\chi}$ simplifies if we consider the case where $\vec{B} = 0$. In that case, there is a range of temperatures where the solution of the equation is small. Let us assume that

$\vec{\chi}$ has small magnitude and we can Taylor expand the right-hand-side of (29) to get

$$\left(1 - \frac{T_C}{T}\right)\vec{\chi}^a = -\frac{27}{45}\left(\frac{T_C}{T}\right)^3 \chi^a(\vec{\chi})^2 + \dots \tag{31}$$

where

$$T_C = \frac{J}{3k_B} \tag{32}$$

Here, we have used the expansion

$$\coth x - \frac{1}{x} = \frac{1}{3}x - \frac{1}{45}x^3 + \dots$$

Then we see that when $T > T_C$, $\vec{\chi} = 0$ is the unique solution whereas, when $T < T_C$, $\vec{\chi} = 0$ is still a solution but it is a local maximum of $f(\vec{\chi})$. There is a non-zero solution for $\vec{\chi}$ which is the minimum.

14. *Show that the solutions of equation (31) are given by*

$$\vec{m} = \vec{\chi} = \begin{cases} 0 & T > T_C \\ \hat{e}\sqrt{\frac{5}{3}\left(\frac{T}{T_C} - 1\right)} & T < T_C \end{cases} \tag{33}$$

where $\hat{e}$ is an arbitrary unit 3-vector. We see that the solution is indeed of small magnitude when T is sufficiently close to T_C. This justifies the use of the Taylor expansion – the idea is that one can always tune T so that it is close enough to T_C that only the first terms in an expansion of the equation for $\vec{\chi}$ in powers of $|\vec{\chi}|$ matter as far as finding the behaviour of $\vec{m}$ and other quantities at or near the phase transition point. We should also recognize the critical exponent

$$\vec{m} \sim \left(\frac{T_C}{T} - 1\right)^{-\beta'} \quad , \quad \beta' = \frac{1}{2}$$

This critical exponent is equal to what we would expect from mean field theory. As we see shortly, all of the critical exponents of this model are identical to those of mean field theory.

The magnetization is sometimes called the "order parameter" in that it being non-zero tells us that the system is in an "ordered state". In this case it the ordered state is a ferromagnet with spontaneous magnetization given by $\vec{m}$ in the equation above.

15. *Use the above solution for $\vec{\chi}$ to show that, for $T \sim T_C$, the Helmholtz free energy is*

$$F[T,B,N] = \begin{cases} k_B T N \ln(4\pi) & T > T_C \\ k_B T N \ln(4\pi) - \frac{5}{4} k_B T_C N \left(1 - \frac{T_C}{T}\right)^2 + \dots & T > T_C \end{cases} \quad (34)$$

where the ellipses represent contributions which go to zero as $T \to T_C$ faster than $\left(1 - \frac{T_C}{T}\right)^2$.

Again, this is typical mean field theory behaviour. We can work out the remainder of the critical exponents and colclude that this phase transition is described by mean field theory.

4 Mean field theory

We have found a few phase transitions whose critical exponents are defined by mean field theory. However, we have, so far, not given a precise definition of what we mean by this term. In a rough sense, the "mean field approximation" is the substitution of the polarization of a neighbouring spin by the average over all of the spins

$$\hat{n}(x) \cdot \hat{n}(x + \hat{i}) \;\to\; \hat{n}(x) \cdot \left(\frac{1}{N} \sum_y \hat{n}(y) \right)$$

We can, however use a definition of mean field theory which, although not quite as directly related to a mean field, gives us a precise starting point. We could define the "mean field theory" approximation as one where

Mean Field Theory

- For a system with a large number of degrees of freedom, such as the O(3) model, we replace the Boltzmann distribution by a an ansatz for the distribution function which is a product over individual probability distributions, one for each spin,

  $$\rho = \prod_x \rho_x(\vec{\chi}_x) \quad (35)$$

  where each factor in the distribution is itself properly normalized

  $$\int [d\vec{\chi}] \rho_x(\vec{\chi}) = 1$$

- We adjust the product distribution so that it minimizes the Helmholtz free energy.

In mean field theory we would compute the magnetization density as

$$< \vec{m}_x >= \int [d\vec{\chi}] \vec{\chi} \rho_x(\chi) \tag{36}$$

and connected correlation functions vanish.

16. *Show that the internal energy, the entropy and the Helmholtz free energy are given by the expressions*

$$U = -J \sum_{x,i} \left[\int [d\vec{\chi}] \vec{\chi} \rho_x(\vec{\chi}) \right] \cdot \left[\int [d\vec{\chi}] \vec{\chi} \rho_{x+\hat{i}}(\vec{\chi}) \right] - \vec{B} \cdot \sum_x \left[\int [d\vec{\chi}] \vec{\chi} \rho_x(\vec{\chi}) \right] \tag{37}$$

$$S = -k_B \sum_x \int [d\vec{\chi}] \rho_x(\vec{\chi}) \ln[\rho_x(\vec{\chi})] \tag{38}$$

$$F = U - TS \tag{39}$$

$$= -J \sum_{x,i} \left[\int [d\vec{\chi}] \vec{\chi} \rho_x(\vec{\chi}) \right] \cdot \left[\int [d\vec{\chi}] \vec{\chi} \rho_{x+\hat{i}}(\vec{\chi}) \right] - \vec{B} \cdot \sum_x \left[\int [d\vec{\chi}] \vec{\chi} \rho_x(\vec{\chi}) \right]$$
$$+ k_B T \sum_x \int [d\vec{\chi}] \rho_x(\vec{\chi}) \ln[\rho_x(\vec{\chi})] \tag{40}$$

We could adjust the function $\rho_x(\vec{\chi})$ to minimize the free energy. We begin by rewriting the free energy as

$$F = \frac{J}{2} \sum_{x,\hat{i}} \left(\int [d\vec{\chi}] \vec{\chi} \rho_{x+\hat{i}}(\vec{\chi}) - \int [d\vec{\chi}] \vec{\chi} \rho_x(\vec{\chi}) \right)^2$$
$$- JD \sum_x \left[\int [d\vec{\chi}] \vec{\chi} \rho_x(\vec{\chi}) \right]^2 - \vec{B} \cdot \sum_x \left[\int [d\vec{\chi}] \vec{\chi} \rho_x(\vec{\chi}) \right]$$
$$+ k_B T \sum_x \int [d\vec{\chi}] \rho_x(\vec{\chi}) \ln[\rho_x(\vec{\chi})] \tag{41}$$

The first term in the above functional is positive definite and the free energy will be minimal only when that term is zero, that is, when

$\int[d\vec{\chi}]\vec{\chi}\rho_x(\vec{\chi})$ does not depend on x. If we assume that this is so, the remainder of the free energy is a sum of independent terms, one for each lattice site. We must minimize them one at a time and the mathematical problem of finding the minimum of each is the same for all values of x. Of course, these are identical minimization problems. They should therefore be minimal for density $\rho(\vec{x})$ which does not depend on x. What remains is

$$F = N\left\{-JD\left[\int[d\vec{\chi}]\vec{\chi}\rho(\vec{\chi})\right]^2 - \vec{B}\cdot\left[\int[d\vec{\chi}]\vec{\chi}\rho(\vec{\chi})\right] + k_B T\int[d\vec{\chi}]\rho(\vec{\chi})\ln[\rho(\vec{\chi})]\right\} + \alpha\left[\int[d\vec{\chi}]\rho(\vec{\chi}) - 1\right] \tag{42}$$

where we have introduced a Lagrange multiplier to enforce the normalization of the density.

To implement the variational principle, we replace $\rho(\vec{\chi})$ by $\rho(\vec{\chi})+\delta\rho(\vec{\chi})$ and we study the term in the functional which is linear in $\delta\rho(\vec{\chi})$.

$$\delta\tilde{F} = N\int[d\vec{\chi}]\delta\rho(\vec{\chi})\left\{-2JD\vec{m}\cdot\vec{\chi} - \vec{B}\cdot\vec{\chi} + k_B T\ln\rho(\vec{\chi}) + \alpha + k_B T\right\} \tag{43}$$

and

$$\rho(\vec{\chi}) = \exp\left(\frac{2JD\vec{m}\cdot\vec{\chi} + \vec{B}\cdot\vec{\chi} - \alpha - k_B T}{k_B T}\right) \tag{44}$$

We determine α by normalizing the distribution

$$\rho(\vec{\chi}) = \frac{\exp\left(\frac{2JD\vec{m}\cdot\vec{\chi}+\vec{B}\cdot\vec{\chi}}{k_B T}\right)}{4\pi\frac{\sinh\frac{|2JD\vec{m}+\vec{B}|}{k_B T}}{\frac{|2JD\vec{m}+\vec{B}|}{k_B T}}} \tag{45}$$

Then, the magnetization is defined as the first moment of the distribution which in turn depends on the magnetization. We then determine the magnetization by self-consistency, that is, it must obey the equation

$$\vec{m} = \int[d\vec{\chi}]\vec{\chi}\frac{\exp\left(\frac{2JD\vec{m}\cdot\vec{\chi}+\vec{B}\cdot\vec{\chi}}{k_B T}\right)}{4\pi\frac{\sinh\frac{|2JD\vec{m}+\vec{B}|}{k_B T}}{\frac{|2JD\vec{m}+\vec{B}|}{k_B T}}} \tag{46}$$

17. *Show that, upon taking the integral in equation (46) we obtain the same equation for the magnetization as we found for the infinite range O(3) model quoted in equation (30) if we redefine the parameter $J \to 2DJ$.*

We could continue to compute the free energy and study its critical behaviour. What we will find is a mirror of what we have already found for the long ranged Ising and O(3) models. In fact, the obervation above, that the magnetization obeys the same equation in the mean feld theory approximation to the O(3) model as it did in the infinite range O(3) model is basically already enough to establish that the critical behaviour that we will find by applying mean field theory to the O(3) model will be identical to that which we have already found in the long-ranged O(3) model. In particular, the critical exponents will be the same. That particular set of critical exponents are called the "critical exponents of mean field theory".