**Problem 1** (5 points): In class we introduced the Hermitian adjoint $X^\dagger$ of an operator $X$ through the dual correspondence
\[ X|\alpha\rangle \leftrightarrow \langle \alpha|X^\dagger, \ \forall |\alpha\rangle \in \mathcal{H}. \] (1)
Denote by $[X]$ the matrix corresponding to the operator $X$ in some ONB $B$, i.e. $[X]_{ij} = \langle a_i|X|a_j\rangle$ for $|a_i\rangle, |a_j\rangle \in B$.

(a) Show that for the matrix representation $[X]$ of a linear operator $X$, taking the Hermitian adjoint according to Eq. (1) amounts to the familiar Hermitian transpose, i.e., $[X^\dagger]_{kl} = ([X]_{lk})^\ast$.

(b) Show explicitly that taking the Hermitian adjoint of an operator $X$ in some matrix representation $[X]$ is basis-independent. That is, consider the transformation of $[X]$ from one ONB to another, and demonstrate that (i) first taking the Hermitian transpose and then transforming to the other ONB and (ii) first transforming to the other ONB and then taking the Hermitian transpose yield the same result.

(c) Likewise, show that taking products of operators in some matrix representation corresponding to an ONB is basis-independent.

**Solution:** (a) We use the property of the scalar product $\langle \gamma|\beta\rangle = \langle \beta|\gamma\rangle$ and the dual correspondence Eq. (1). Choosing $|\gamma\rangle = X|\alpha\rangle$, we find
\[ \langle \alpha|X^\dagger|\beta\rangle = \left(\langle \alpha|X^\dagger\right)|\beta\rangle = ([\langle \beta|X|\alpha\rangle])^\ast = \langle \beta|X|\alpha\rangle^\ast. \]
Now choose $|\alpha\rangle, |\beta\rangle$ in some ONB. Then,
\[ [X^\dagger]_{\alpha\beta} = ([X]_{\beta\alpha})^\ast. \]

(b) The transformation from one ONB to another is by conjugation under some unitary matrix $U$,
\[ [X] \rightarrow [X'] = U[X]U^\dagger. \]
Then, $([X']^\dagger = (U[X]U^\dagger)^\dagger = U[X]^\dagger U^\dagger$. That is, first transforming and then taking the Hermitian transpose is the same as first taking the Hermitian transpose and then transforming.

(c) $[X'][Y]' := [U[X]^\dagger U[Y]^\dagger = U[X][Y]^\dagger =: [XY]'$. The second equality follows from $U^\dagger U = I$ ($U$ is unitary).

**Problem 2** (5 points): The Hamiltonian operator for a 2-state system is given by
\[ H = a \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right), \] (2)
where $a$ is a constant with the dimension of energy, and the kets $|1\rangle$ and $|2\rangle$ form an orthonormal basis. Find the energy eigenvalues and the corresponding eigenkets, as linear combinations of $|1\rangle$ and $|2\rangle$.

**Solution:** Upon the identification $|1\rangle \cong \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ and $|2\rangle \cong \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$, the Hamiltonian $H$ can be represented by a $2 \times 2$ matrix
\[ H \cong a \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right). \] (3)
Therefore, consider a spin-1 particle which can assume the states problem 4.

Problem 3. (5 points) The Pauli matrices $\sigma_x$, $\sigma_y$, and $\sigma_z$ are 2×2 matrices defined as follows:

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Consider a vector $n = (n_x, n_y, n_z)^T$ of unit length, and $n \cdot \vec{\sigma} := n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$. Are there $r \in \mathbb{R}$ such that

$$O = I + r n \cdot \vec{\sigma}$$

is a projector?

Solution: For $O$ to be a projector we require $O^\dagger = O$ and $O^2 = O$. The first property holds for all $r \in \mathbb{R}$. Regarding the second property, we find just by expanding

$$(n \cdot \vec{\sigma})^2 = I, \forall n \text{ with } |n| = 1.$$

Therefore,

$$O^2 = \frac{(1 + r^2)I + 2rn \cdot \vec{\sigma}}{4},$$

and $O^2 = O$ iff:

$$r = \pm 1.$$  

Problem 4. (5 points) Consider a spin-1 particle which can assume the states

$$|+\rangle := |s = 1, s_z = 1\rangle, \quad |0\rangle := |s = 1, s_z = 0\rangle, \quad |−\rangle := |s = 1, s_z = −1\rangle,$$

and any linear combination thereof. The $S_z$-operator for this system is $S_z = \hbar(|+\rangle\langle+| - |−\rangle\langle−|$. Consider the sequential measurement of the operator $(S_z)^2$ (first) and $S_z$ (second) upon the initial state

$$|\psi\rangle = \frac{|+\rangle + |0\rangle + |−\rangle}{\sqrt{3}}.$$

(a) What are the possible measurement outcomes of $(S_z)^2$ and the probabilities of obtaining them?

(b) What are the possible measurement outcomes for $S_z$ in the second measurement, and the conditional probabilities for obtaining them? (The outcome probabilities are conditioned upon the outcome of the first measurement.)

Solution: (a) $S_z^2 = |+\rangle\langle+| + |−\rangle\langle−|$, which has eigenvalues 0, 1. Thus, the possible outcomes of an $S_z^2$-measurement of a spin-1 particle are 0 and 1. The projectors $P_{S_z,0}$ and $P_{S_z,1}$ on the sub-spaces corresponding to the eigenvalues 0 and 1, respectively, are

$$P_{S_z,0} = |0\rangle\langle0|, \quad P_{S_z,1} = |+\rangle\langle+| + |−\rangle\langle−|.$$  

Then, the probabilities $p_{S_z,0}$ and $p_{S_z,1}$ for obtaining outcome 0 or 1, respectively, are given by the Born rule

$$p_{S_z,0} = \langle\psi|P_{S_z,0}\psi\rangle = \frac{1}{3}, \quad p_{S_z,1} = \langle\psi|P_{S_z,1}\psi\rangle = \frac{2}{3}.$$  

(b) The possible measurement outcomes of the $S_z$-measurement are ±1. The corresponding projectors are

$$P_{S_z,1} = |+\rangle\langle+|, \quad P_{S_z,0} = |0\rangle\langle0|, \quad P_{S_z,−1} = |−\rangle\langle−|.$$
Case I: The outcome of the $S^2_z$-measurement was 1. Then, the post-measurement state is

$$|\tilde{\psi}(1)\rangle = \frac{P_{S^2_z,1}|\psi\rangle}{\sqrt{\langle \psi |P_{S^2_z,1}|\psi\rangle}} = \frac{|+\rangle + |-\rangle}{\sqrt{2}}.$$  

By the Born rule, the conditional probabilities $p_{S^2_z,1}$, $p_{S^2_z,0}$, $p_{S^2_z,-1}$ for obtaining the outcomes 1,0,-1, respectively, are

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<tr>
<th>$p_{S^2_z,1}$</th>
<th>$p_{S^2_z,0}$</th>
<th>$p_{S^2_z,-1}$</th>
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<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
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Case II: The outcome of the $S^2_z$-measurement was 0. Then, the post-measurement state is

$$|\tilde{\psi}(0)\rangle = \frac{P_{S^2_z,0}|\psi\rangle}{\sqrt{\langle \psi |P_{S^2_z,0}|\psi\rangle}} = |0\rangle.$$  

By the Born rule, the conditional probabilities $p_{S^2_z,1}$, $p_{S^2_z,0}$, $p_{S^2_z,-1}$ for obtaining the outcomes 1,0,-1, respectively, are

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<td>0</td>
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Problem 5 (5 points): A spin 1/2 system is known to be in an eigenstate of $\mathbf{S} \cdot \mathbf{n}$ with eigenvalue $\hbar/2$. The unit vector $\mathbf{n}$ lies in the $xz$-plane at an angle $\phi$ with the positive $z$-axis, i.e. $\mathbf{S} \cdot \mathbf{n} = n_x(\phi)S_x + n_z(\phi)S_z$. Furthermore, the angle $\phi$ is oriented such that for $\phi = \pi/2$, $\mathbf{S} \cdot \mathbf{n} = S_x$.

(a) Suppose $S_z$ is measured. What is the probability for finding $+\hbar/2$?
(b) Evaluate the dispersion in $S_z$, that is, $\langle (S_z - \langle S_z \rangle)^2 \rangle$. Check your answers for the special cases of $\phi = 0$, $\phi = \pi/2$ and $\phi = \pi$.

Solution: (a) With the specifications given, $\mathbf{n} \cdot \mathbf{\hat{S}} = \cos \phi \mathbf{\hat{S}}_z + \sin \phi \mathbf{\hat{S}}_x$. The eigenvector with eigenvalue $+\hbar/2$ is

$$|\phi, +\rangle = \cos \frac{\phi}{2}|+\rangle + \sin \frac{\phi}{2}|-\rangle,$$

where $|+\rangle$, $|-\rangle$ are the eigenstates of $\mathbf{\hat{S}}_x$. (To check the above, you may find the relations $\sin 2\alpha = 2\sin \alpha \cos \alpha$, $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$ useful.) The probability of obtaining the outcome $+\hbar/2$ in a measurement of $\mathbf{\hat{S}}_z$ on $|\phi, +\rangle$ is $p(+) = |\langle x, + |\phi, + \rangle|^2$, where $|x, +\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$. Thus,

$$p(+) = \frac{1}{2} \left| \cos \frac{\phi}{2} + \sin \frac{\phi}{2} \right|^2 = \frac{1+\sin \phi}{2}.$$  

(11)

(b) Then, the probability $p(-)$ for obtaining the outcome $-\hbar/2$ in the $\mathbf{\hat{S}}_x$-measurement is $p(-) = 1 - p(+) = (1 - \sin \phi)/2$. The expectation value of $\mathbf{\hat{S}}_x$ therefore is

$$\langle \mathbf{\hat{S}}_x \rangle = \frac{\hbar}{2} \sin \phi.$$  

The dispersion (variance) of $\mathbf{\hat{S}}_x$ is $\text{Var}(\mathbf{\hat{S}}_x) = \langle (\mathbf{\hat{S}}_x - \langle \mathbf{\hat{S}}_x \rangle)^2 \rangle = \langle \mathbf{\hat{S}}^2_x \rangle - \langle \mathbf{\hat{S}}_x \rangle^2$. Also, $\mathbf{\hat{S}}^2_x = \frac{\hbar^2}{4} \mathbf{I}$, and thus

$$\text{Var}(\mathbf{\hat{S}}_x) = \frac{\hbar^2}{4} \cos^2 \phi.$$  

(12)

For angles $\phi = 0$, $\pi/2$, $\pi$ the variance is $\hbar^2/4$, 0, $\hbar^2/4$, respectively. This agrees with the expectation, since for $\phi = 0$, $\pi$ the measured state is an eigenstate of $\mathbf{S}_z$, and for $\phi = \pi/2$ an eigenstate of $\mathbf{S}_x$.

Problem 6 (5 points). A beam of spin 1/2 atoms goes through a series of Stern-Gerlach-type measurements as follows:

(a) The first measurement accepts $s_z = \hbar/2$ atoms and blocks $s_z = -\hbar/2$ atoms.

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(b) The second measurement accepts \( s_n = \hbar/2 \) atoms and blocks \( s_n = -\hbar/2 \) atoms, where \( n \) is a unit vector lying in the \( xx \)-plane, at an angle \( \beta \) with the positive \( z \)-axis.

(c) The third measurement accepts \( s_z = -\hbar/2 \) atoms and blocks \( s_z = \hbar/2 \) atoms.

What is the intensity of the final \( s_z = -\hbar/2 \) beam when the \( s_z = \hbar/2 \) beam surviving the first measurement is normalized to unity? How must the second apparatus be oriented (i) to maximize, and (ii) to minimize the intensity of the final \( s_z = -\hbar/2 \) beam?

**Solution:** The state of the atom after passing the first SG apparatus is \( |+\rangle \), the state after passing the second apparatus is \( |\beta, +\rangle = \cos \frac{\beta}{2} |+\rangle + \sin \frac{\beta}{2} |\rangle \), (see Problem 4), and the state after passing the third apparatus is \( |\rangle \). The probability for the atom to pass the second SG apparatus (conditioned upon having passed the first one) is \( p_{pass,2} = |\langle + | \beta, + \rangle|^2 = \cos^2 \frac{\beta}{2} \). Likewise, the probability for the atom to pass the third SG apparatus (conditioned upon having passed the second one) is \( p_{pass,3} = |\langle \beta, + | \rangle|^2 = \sin^2 \frac{\beta}{2} \). The two events are independent, and therefore the probability \( p_{pass,23} \) for passing both SG apparatus is

\[
p_{pass,23} = \cos^2 \frac{\beta}{2} \sin^2 \frac{\beta}{2} = \frac{1}{4} \sin^2 \beta.
\]

The final intensity is the initial intensity \( \times p_{pass,23} \). The output intensity is maximized for \( \beta = \pm \frac{\pi}{2} \) and minimized for \( \beta = 0, \pi \).

**Problem 7** (5 points). (Continuation from class) We noted the Born rule for measurement for the general (potentially degenerate) case. If \( P_a \) is the projector on the subspace corresponding to the measurement outcome \( a \) for the observable \( A \), then the probability \( p(a) \) for obtaining the outcome \( a \) given the quantum state \( |\Psi\rangle \) is

\[
p(a) = \langle \Psi | P_a | \Psi \rangle.
\]

Show that \( p(a) \in \mathbb{R} \) and \( p(a) \geq 0 \) for all possible measurement outcomes \( a \), and that \( \sum_a p(a) = 1 \).

**Solution.** \( p(a) = \langle \Psi | P_a | \Psi \rangle = \langle \Psi | P^2_a | \Psi \rangle = \langle \Psi | P^3_a | \Psi \rangle = \langle \Psi | P^4_a | \Psi \rangle = p^4(a) \). Hence, \( p(a) \in \mathbb{R} \). Furthermore, \( p(a) := \langle \Psi | P_a | \Psi \rangle = \langle \Psi | P^2_a | \Psi \rangle = \langle \Psi | P^3_a | P | \Psi \rangle = \left( \langle \Psi | P^3_a \right) \left( P | \Psi \rangle \right) \). Hence, \( p(a) \geq 0 \).

Finally, denote by \( P_a \) the projector on the subspace of eigenstates of \( A \) with eigenvalue \( a \), \( P_a = \sum_i |a, i \rangle \langle a, i | \) where \( A |a, i \rangle = a |a, i \rangle \). The key point is that \( E_A := \{|a, i \rangle, \forall a, i \} \) is an ONB of \( \mathcal{H} \). You may use this fact without proof, since its pure linear algebra, but a proof is given below. Then, \( \sum_a |a, i \rangle \langle a, i | = I \), and hence \( \sum_a P_a = I \). Therefore, \( \sum_a p(a) = \sum_a \langle \Psi | P_a | \Psi \rangle = 1 \).

Why \( E_A \) is a basis of \( \mathcal{H} \), for any observable \( A \): First, \( \mathcal{H} \) is by definition complete, hence has a basis. Then, it also has an ONB, \( B_0 \), which can be constructed e.g. by the Gram-Schmidt procedure. We may compute the matrix \([A]\) corresponding to \( A \) in the basis \( B_0 \). Now, \([A]\) may be brought into Jordan normal form \([J]\) by a unitary transformation \( U \), \([J] = U\,[A]\,U^\dagger \). Since \([A]\) is a matrix over the complex numbers, the Jordan normal form always exists. Furthermore, since \( A \) is Hermitian, \([A]\) is a Hermitian matrix, and so is \([J]\). Therefore, \([J]\) is actually a diagonal matrix. Its eigenvectors \( e_j \) span the full space. Therefore, the eigenvectors of \([A]\), \([U^\dagger \, e_j]\) also span the full space. \([U^\dagger \, e_j]\) corresponds to \( E_A \) in the matrix representation induced by \( B_0 \).

Total: 35 points.