Physics 503 Problem Set 6 Solutions

1a) There are again two types of possible interactions, F type and V type, in the notation of Shankar. Following Shankar, there is no renormalization of the F-type interactions, due to a factor of $(d\Lambda)^2$. Thus we only need to focus on the V type, involving a pair of ψ operators at points $\vec{K}_1$ and $-\vec{K}_2$ and a pair of $\bar{\psi}$ operators at points $\vec{K}_3$ and $-\vec{K}_4$. These are the interactions indicated schematically in the problem set. In general we could write the interaction as $V_{\alpha\beta\gamma\delta}$ but the form of this 4-index tensor is restricted by $SU(2)$ spin rotation invariance. With 4 spin-1/2 indices there are two ways of making a spin singlet. Either we combine α and β to make a spin singlet or triplet. Likewise we can combine γ and δ to make singlet or triplet. To get an overall spin singlet we must either form singlets from both the ψ pair and $\bar{\psi}$ pair, corresponding to V_s, or else make triplets from both ψ pair and $\bar{\psi}$ pairs. We must then contract the two triplets to form a singlet, corresponding to V_t. To see that the V_t term corresponds to a product of triplets, consider how

$$\sum_{\gamma\delta} \psi_\gamma (\vec{\sigma}\sigma_2)_{\gamma\delta}\psi_\delta \tag{1}$$

transforms under a spin rotation:

$$\psi_\gamma \to \sum_{\gamma\delta} \left[e^{i\vec{\theta}\cdot\vec{\sigma}}\right]_{\gamma\delta} \psi_\delta. \tag{2}$$

Using the property of Pauli matrices:

$$\sigma_2\sigma_a = -\sigma_a^\dagger \sigma_2 \tag{3}$$

we see that:

$$e^{i\vec{\theta}\cdot\vec{\sigma}}\vec{\sigma}\sigma_2 e^{i\vec{\theta}\cdot\vec{\sigma}} = e^{i\vec{\theta}\cdot\vec{\sigma}}\vec{\sigma}e^{-i\vec{\theta}\cdot\vec{\sigma}}\sigma_2. \tag{4}$$

We now use the fact that this gives the spin-1/2 representation of the rotation group: $R(\vec{\theta})\vec{\sigma}\sigma_2$.

$$e^{i\vec{\theta}\cdot\vec{\sigma}}\sigma_a e^{-i\vec{\theta}\cdot\vec{\sigma}} = \sum_b R_{ab}(\vec{\theta})\sigma_b \tag{5}$$

where $R(\vec{\theta})$ is the rotation corresponding to $\vec{\theta}$. For example $R(0,0,\theta_z)$ is a rotation by $-2\theta_z$ around the z-axis. Thus the V_t term in the action is indeed rotationally invariant.

b) Noting that the 3 matrices $(\sigma_a\sigma_2)$ are all symmetric, we see that

$$V_t(\theta_{13}+\pi) = -V_t(\theta_{13}) \tag{6}$$

just as in the spinless case, as discussed by Shankar on page 167. On the other hand, since $\epsilon_{\alpha\beta}$ is anti-symmetric, V_s must have the opposite symmetry:

$$V_s(\theta_{13}+\pi) = V_s(\theta_{13}). \tag{7}$$

2a) I begin with the normalization as in Shankar (251) generalized to 2D:

$$\psi_j = \int \frac{d^2K}{(2\pi)^2} e^{-i\vec{K}\cdot\vec{r}_j}\psi(\vec{K}) \tag{8}$$

where $\vec{r}_j$ is the location of the jth lattice site. I then use:

$$\sum_j e^{i(\vec{K}_1+\vec{K}_2-\vec{K}_3-\vec{K}_4)\cdot\vec{r}_j} = N\delta_{\vec{K}_1+\vec{K}_2,\vec{K}_3+\vec{K}_4} \tag{9}$$

where N is the number of lattice sites. Taking the area of the lattice to infinity, this becomes:

$$\sum_j e^{i(\vec{K}_1+\vec{K}_2-\vec{K}_3-\vec{K}_4)\cdot\vec{r}_j} = (2\pi)^2\delta^2(\vec{K}_1+\vec{K}_2-\vec{K}_3-\vec{K}_4). \tag{10}$$

The pairing part of the Hubbard model, for small K_F, in the low energy limit is then:

$$S_{int} = UK_F \int_{K\omega\theta} \bar{\psi}_\uparrow(4)\bar{\psi}_\downarrow(3)\psi_\uparrow(2)\psi_\downarrow(1). \tag{11}$$

(The factor of K_F arises from three factors of K_F from the three integrals over d^2K_i after adsorbing a factor of $\sqrt{K_F}$ into each fermion field.) Thus we see that the V functions have no dependence on θ_{13} in this case of purely on-site interactions. This is simpler than the case of nearest neighbour interactions discussed in Shankar. We may now make the replacement:

$$\psi_\uparrow(2)\psi_\downarrow(1) \to (1/2)[\psi_\uparrow(2)\psi_\downarrow(1) - \psi_\downarrow(2)\psi_\uparrow(1)] \tag{12}$$

by switching the 1 and 2 integration variables and using the anti-commutation relations. Doing the same thing with the $\bar{\psi}\bar{\psi}$ factor, we conclude that:

$$V_s(\theta_{13}) = UK_F/4 \tag{13}$$
$$V_t = 0. \tag{14}$$

b) The RG equation is derived as in Shankar for the spinless case, except for some factors of 2 and minus signs. Note that the RG equation for the spinless case in (381)of Shankar actually involves the sum of two terms:

$$\frac{dV(\theta_1 - \theta_3)}{dt} = -\frac{1}{16\pi^2}\int_0^{2\pi} \frac{d\theta}{2\pi}[V(\theta_1 - \theta)\left[V((\theta - \theta_3) - V(\theta + \pi - \theta_3)\right]. \tag{15}$$

These two terms arise from the two ways of contracting the two ψ operators from the first vertex with the two $\bar{\psi}$ operators from the second. The relative minus sign arises from Fermi statistics. The shift by π in the second term is due to the fact that, in the BCS interaction, the two ψ operators are at opposite points on the Fermi surface (with angles differing by π). The reason these two terms add rather than cancel is due to the symmetry property: $V(\theta_{13} + \pi) = -V(\theta_{13})$. The RG equation for V_s is:

$$\frac{dV_s(\theta_1 - \theta_3)}{dt} = -\frac{2!2!}{16\pi^2}\int_0^{2\pi} \frac{d\theta}{2\pi}[V_s(\theta_1 - \theta)\left[V_s((\theta - \theta_3)\sum_{\alpha\beta}\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} - V_s(\theta + \pi - \theta_3)\sum_{\alpha\beta}\epsilon_{\alpha\beta}\epsilon_{\beta\alpha}\right]. \tag{16}$$

The factor of (2!2!) arises because of the fact that in the definition of quartic interaction in Shankar Eq. (283) the action is defined with a factor of $\frac{1}{2!2!}$, while this was not done in the statement of this problem. The sum over spin factors gives 2 in the first term and -2 in the second. The symmetry property, Eq. (7) again insures that the two terms add rather than cancelling. Thus:

$$\frac{dV_{sl}}{dt} = -\frac{2V_{sl}^2}{\pi}. \tag{17}$$

Here V_{ls} is the Fourier transforms of $V_s(\theta)$ normalized as in Shankar (382). The calculation of the β-function for V_s is similar. In this case we must use:

$$\mathrm{tr}(\sigma_2\sigma_a\sigma_b\sigma_2) = \mathrm{tr}[\sigma_2\sigma_a(\sigma_b\sigma_2)^T] = 2\delta_{ab}. \tag{18}$$

Now the spin factor has the same sign for both terms but the symmetry property of V_t, Eq. (6), again insures that the two terms add. Due to the minus sign in the statement of the problem, we now find:

$$\frac{dV_{tl}}{dt} = \frac{2V_{tl}^2}{\pi}. \tag{19}$$

No mixing of V_s and V_t occurs, as expected from their different symmetry and as follows since

$$\sum_{\alpha\beta}\epsilon_{\alpha\beta}(\sigma_a\sigma_2)_{\beta\alpha} = 0 \tag{20}$$

due to the fact that $\sigma_a\sigma_2$ is a symmetric matrix.