Physics 503 Problem Set 5
Solutions

1. If we raise fermions from the highest n_m occupied levels in the ground state by m steps this increases the energy by $(2\pi/L)n_m m$ since it is equivalent to raising each of n_m fermions by m steps. (See the figure for an example.) Note that it is possible to make this move without double occuping any levels, which would be disallowed by the Pauli principle. The resulting state now has m unoccupied levels below the cluster of n_m occupied ones. Now we may safely raise n_{m-1} fermions from below these unoccupied levels up by $m-1$ steps partially filling in the gap from the first move. This never violates the Pauli principle due to to the available m unoccupied levels. Next we may move n_{m-2} fermions by $m-2$ levels, etc. until finally moving n_1 fermions up only one level. This construction always gives an allowed state (not violating the Pauli principle). The starting point $m \geq 0$ is arbitrary. Summing up the energy from each "move", we see that the total energy is:

$$E - E_0 = (2\pi/L)\sum_{m=1}^{\infty} m n_m. \tag{1}$$

It can also be seen to give the most general multiple particle-hole excited state for the right branch only. This follows because this construction can give an arbitrary pattern of occupied and unoccupied levels. An arbitrary pattern consists, counting down from the highest occupied state, of some number of occupied states, followed by an arbitrary number of unoccupied states, followed by an arbitrary number of occupied states, etc. In the above construction, the number of occupied levels at the top is n_p where p is the largest m for which $n_m \neq 0$. The size of the first gap is $p - p' - 1$ where p' is the next highest value of m for which $n_m \neq 0$. The number of occupied states in the next group is $n_{p]}$, etc.

b) In the bosonic theory, the harmonic oscillator energies are $2\pi m/L$ for $n \geq 1$. The energy of a state with n_m bosons in the m^{th} level is precisely Eq. (**??**).

2a) Writing the normalization factor as:

$$\psi_L = A e^{\pm i\sqrt{4\pi}\phi_R} \tag{2}$$

we have

$$\psi_R(x)\psi_R(y) = A^2 e^{i\sqrt{4\pi}\phi_R(x)} e^{i\sqrt{4\pi}\phi_R(y)}. \tag{3}$$

Using the Baker-Haussdorf identity this becomes:

$$\begin{aligned}\psi_R(x)\psi_R(y) &= A^2 e^{i\sqrt{4\pi}(\phi_R(x)+\phi_R(y))} e^{-2\pi[\phi_R(x),\phi_R(y)]} \\ &= A^2 e^{i\sqrt{4\pi}(\phi_R(x)+\phi_R(y))} e^{(-i\pi/2)\text{sgn}(x-y)} = -i\cdot\text{sgn}(x-y)A^2 e^{i\sqrt{4\pi}(\phi_R(x)+\phi_R(y))}.\end{aligned} \tag{4}$$

(Here I used the fact that sgn $(x-y)$ is an integer.) Multiplying the operators in the opposite order just corresponds to switching $x \leftrightarrow y$, giving

$$\psi_R(y)\psi_R(x) = -i\cdot\text{sgn}(y-x)A^2 e^{i\sqrt{4\pi}(\phi_R(x)+\phi_R(y))}. \tag{5}$$

We see that indeed:

$$\psi_R(y)\psi_R(x) = -\psi_R(x)\psi_R(y). \tag{6}$$

b) The only reasonable choice is

$$[\phi_R(x), \phi_L(y)] = c, \tag{7}$$

for some conveniently chosen constant, c. This is required so that $[J_R, J_L] = 0$ and, also $[J_R, \psi_L] = 0$. Using again the Baker-Haussdorf identity:

$$\psi_R(x)\psi_L(y) = A^2 e^{i\sqrt{4\pi}(\phi_R(x)-\phi_L(y))} e^{i2\pi c}. \tag{8}$$

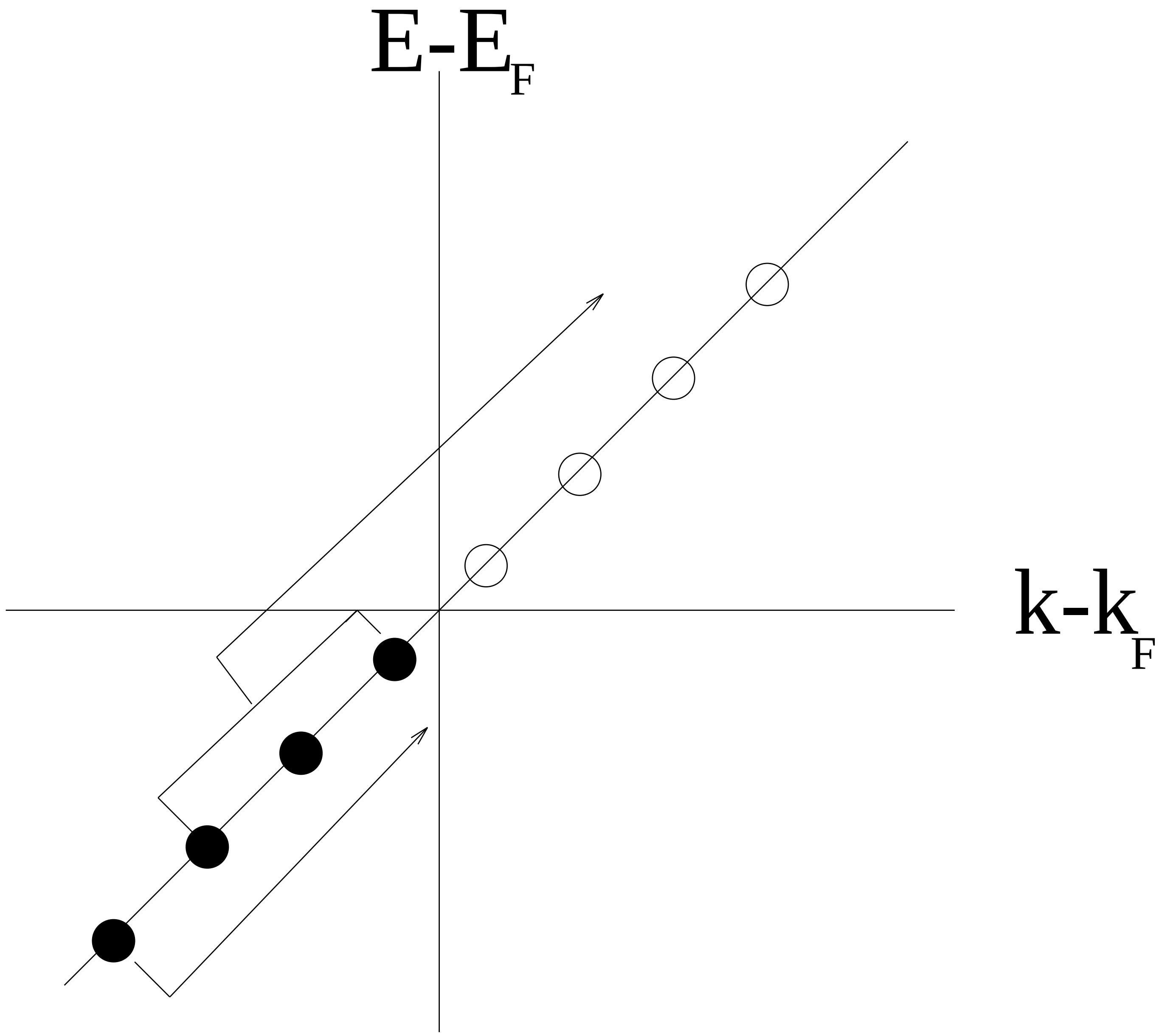

FIG. 1: Example of an excitation in which the fermions in the highest 3 occupied levels are raised 4 steps, corresponding to $n_3 = 4$.

On the other hand:

$$\psi_L(y)\psi_R(x) = A^2 e^{i\sqrt{4\pi}(\phi_R(x)-\phi_L(y))} e^{-i2\pi c}. \tag{9}$$

Choosing $c = 1/4$, we have:

$$e^{\pm i2\pi c} = \pm i \tag{10}$$

giving the desired anti-commutation relations.