Physics 503 Problem Set 5

Due Thursday, March 29

1. In this problem you are to show that the finite size spectra, with periodic boundary conditions, are the same for a free boson and free fermion field theory. Assume a system size, l, so that the allowed wave-vectors are $k_n = 2\pi n/l$. Also assume a linear dispersion relation, and set the velocity, $v_F = 1$, so that the single fermion (or boson) energies are $E_n = |k_n|$.

a) Derive a formula for a general multiple particle-hole excitation in the fermionic theory, considering only the right-moving sector. Do this by proving that an arbitrary multiple particle-hole excitation can be constructed by raising the first n_m electrons immediately below the Fermi energy by m levels, then raising the next n_{m-1} electrons by $m-1$ levels, ... and finally raising the next n_1 electrons by 1 level. (m can be an arbitrary positive integer.) Write a sum formula for the energy of this state.

b) Show that this state has the same energy as a state in the free boson theory and identify what this state is by specifying the occupation numbers for the various (right-moving) single-boson energy levels.

c) Argue that there is a one-to-one correspondence between all particle-hole states and all boson states.

Comments: Actually, due to the multiple degeneracy of the spectrum, there are many possible ways of identifying fermionic and bosonic spectra. Any simple one will suffice to solve this problem. The above correspondence can actually be extended to include fermionic states where extra electrons are added or subtracted from the ground state. These correspond to soliton-type states in the boson theory. However, I won't ask you to show that here.

2a) Using the bosonization formula:

$$\psi_R(t-x) \propto e^{-i\sqrt{4\pi}\phi_R(t-x)} \tag{1}$$

and the equal-time boson commutation relations derived in class:

$$[\phi_R(x), \phi_R(y)] = \frac{i}{4}\mathrm{sgn}(x-y), \tag{2}$$

show that:

$$\psi_R(x)\psi_R(y) = -\psi_R(y)\psi_R(x) \tag{3}$$

regardless of the sign of $x-y$.

Hint: The Baker-Haussdorf theorem:

$$e^A e^B = e^{A+B} e^{1/2[A,B]} \tag{4}$$

for any two operators A and B whose commutator is a c-number (commutes with A and B) is useful here.

b) Guess an equal time commutation relation $[\phi_L(x), \phi_R(y)]$, which satisfies the two requirements of bosonization: $\{\psi_L(x), \psi_R(y)\} = 0$ and $[J_L(x), J_R(y)] = 0$ together with the bosonization dictionary:

$$\psi^\dagger_{R,L} \propto e^{\pm i\sqrt{4\pi}\phi_{R,L}}$$
$$J_{R,L} = \pm\frac{1}{\sqrt{4\pi}}\partial_\mp \phi_{R,L}. \tag{5}$$

(It might appear that $\phi_R(x)$ and $\phi_L(y)$ must commute since their mode expansions involve independent creation and annihilation operators. The reason that this is not neccessarily so is because of a possible overlap in the zero momentum modes. If you think this gives you a hint as to what the commutator must be, you are right!)