Physics 503, Problem Set 4, Due in class on Thursday, March 15

1) In this problem you will repeat the RG calculation for the Kondo model, to second order in the Kondo coupling, at finite temperature. We can follow the same procedure, expanding e^S to second order in the part of the interaction term which mixes $>$ Fourier modes with $<$ Fourier modes. Integrating out the $>$ modes in class we had to evaluate the Grassmann average:

$$< \psi(\vec{k},\tau_1)\bar{\psi}(\vec{k},\tau_2) >= [\theta(\tau_1-\tau_2)\theta(\epsilon_k)-\theta(\tau_2-\tau_1)\theta(-\epsilon_k)]\exp[(\tau_2-\tau_1)\epsilon_k] \tag{1}$$

a) Show that, at finite temperature, we get a similar expression except that the factors $\theta(\epsilon_k)$ and $\theta(-\epsilon_k)$ get replaced by functions, $g_1(\beta\epsilon_k)$ and $g_2(\beta\epsilon_k)$ respectively, involving the Fermi function $n_F(\beta\epsilon_k)$ which you are to evaluate.
b) In class, I took the τ_1 and τ_2 integrals to run from $-\infty$ to ∞ but now we should let them both run from 0 to β. After ignoring irrelevant operators, effectively replacing the argument of $\psi_<(\tau_2)$ by τ_1, I was left with evaluating an integral over $\tau_2-\tau_1 \equiv \tau'$:

$$I(\epsilon_k) \equiv \int_{-\infty}^{\infty} d\tau'[\theta(-\tau')\theta(\epsilon_k)-\theta(\tau')\theta(-\epsilon_k)]\mathrm{sgn}(-\tau')\exp(\tau'\epsilon_k). \tag{2}$$

I showed that this integral gives

$$I(\epsilon_k) = \frac{1}{|\epsilon_k|}. \tag{3}$$

Show that, at finite T this gets replaced by:

$$I(\beta,\epsilon_k) \equiv g_1(\beta\epsilon_k)\int_{-\tau_1}^{0} d\tau' \exp(\tau'\epsilon_k) + g_2(\beta\epsilon_k)\int_{0}^{\beta-\tau_1} d\tau' \exp(\tau'\epsilon_k). \tag{4}$$

where g_1 and g_2 are the functions you found in part a). Evaluate the integrals to find the function $I(\beta,\epsilon_k)$ which replaces $1/|\epsilon_k|$. You should find that the sum of integrals in fact doesn't depend on τ_1. Check that your function reduces to $1/|\epsilon_k|$ as $T \to 0$.
c) The next step followed in class was to integrate over ϵ_{k_2} over the $>$ region, $D' < |\epsilon| < D$. Let's assume that D is small compared to the band width so that the density of states is nearly constant over this energy range. Then the integral evaluated in class was proportional to

$$\int_{D'}^{D} \frac{d\epsilon}{\epsilon} = \ln(D/D'). \tag{5}$$

At finite T this gets replaced by:

$$\int_{D'}^{D} d\epsilon I(\beta,\epsilon) \tag{6}$$

where I is the function found in part b). (In both cases we may restrict the integration to $\epsilon > 0$ by inserting a factor of 2 since the function I is even.) Assume that $D \gg T = 1/\beta$ and consider the behavior of this integral as we lower D' from values $\gg T$ to $\ll T$. Show that, when $D' \gg T$ we obtain approximately the $T=0$ result of Eq. (5) and that when $D' \ll T$ we obtain a result which depends only weakly on D'.
d) Using your results from c) argue that the effective coupling, calculated at finite T, when the reduced cut-off is $D \approx T \gg T_K$, is approximately

$$\lambda_{eff}(T) \approx 1/\ln(T/T_K). \tag{7}$$

This was the result I used in class, without justification, to calculate the finite T conductivity.

2) As mentioned in class (without proof) the β-function for the Kondo model, to third order in the dimensionless coupling constant, λ, is:

$$d\lambda/d\ln D = -\lambda^2 + (1/2)\lambda^3. \tag{8}$$

The bare cut-off, D_0, is of order E_F. If we keep only the second order term in the β-function, we obtain:

$$T_K \propto E_F e^{-1/\lambda_0} \tag{9}$$

where $\lambda_0 = J\nu$ is the bare Kondo coupling. Eq. (9) is obtained by determining the energy scale at which the effective Kondo coupling is O(1). By integrating Eq. (1), obtain a more accurate estimate of the Kondo temperature of the form:

$$T_K \propto E_F \lambda_0^p e^{-1/\lambda_0} \tag{10}$$

for some exponent p which you are to calculate.