Physics 503, Problem Set 4 Solutions

a) At finite temperature we have

$$< \psi(\vec{k},\tau_1)\bar{\psi}(\vec{k},\tau_2) > = [\theta(\tau_1-\tau_2) < c_{\vec{k}}(\tau_1)c^\dagger_{\vec{k}}(\tau_2) > -\theta(\tau_2-\tau_1) < c^\dagger_{\vec{k}}(\tau_2)c_{\vec{k}}(\tau_1) >]$$
$$= [\theta(\tau_1-\tau_2) < c_{\vec{k}}c^\dagger_{\vec{k}} > -\theta(\tau_2-\tau_1) < c^\dagger_{\vec{k}}c_{\vec{k}} >]\exp[(\tau_2-\tau_1)\epsilon_k] \quad (1)$$

And by using $< c_{\vec{k}}c^\dagger_{\vec{k}} >= 1-n_F(\beta\epsilon_k)$ and $< c^\dagger_{\vec{k}}c_{\vec{k}} >= n_F(\beta\epsilon_k)$ we get

$$< \psi(\vec{k},\tau_1)\bar{\psi}(\vec{k},\tau_2) > [\theta(\tau_1-\tau_2)(1-n_F(\beta\epsilon_k)) - \theta(\tau_2-\tau_1)n_F(\beta\epsilon_k)]\exp[(\tau_2-\tau_1)\epsilon_k] \quad (2)$$

b) By using the commutation relations of impurity spin operators we get

$$I(\beta,\epsilon_k) = \int_0^\beta d\tau_2[\theta(\tau_1-\tau_2)(1-n_F(\beta\epsilon_k)) - \theta(\tau_2-\tau_1)n_F(\beta\epsilon_k)]\exp[(\tau_2-\tau_1)\epsilon_k]\mathrm{sgn}(\tau_1-\tau_2)$$
$$= (1-n_F(\beta\epsilon_k))\int_0^{\tau_1}\exp[(\tau_2-\tau_1)\epsilon_k]d\tau_2 \ + \ n_F(\beta\epsilon_k)\int_{\tau_1}^{\beta}\exp[(\tau_2-\tau_1)\epsilon_k]d\tau_2$$
$$I(\beta,\epsilon_k) = (1-n_F(\beta\epsilon_k))\int_{-\tau_1}^{0}\exp[\tau\epsilon_k]d\tau \ + \ n_F(\beta\epsilon_k)\int_0^{\beta-\tau_1}\exp[\tau\epsilon_k]d\tau \quad (3)$$

By doing the integration we have

$$I(\beta,\epsilon_k) = \frac{1-n_F(\beta\epsilon_k)}{\epsilon_k}(1-e^{-\tau_1\epsilon_k}) + \frac{n_F(\beta\epsilon_k)}{\epsilon_k}(e^{(\beta-\tau_1)\epsilon_k}-1)$$
$$I(\beta,\epsilon_k) = \frac{1-2n_F(\beta\epsilon_k)}{\epsilon_k} \quad (4)$$

Where the τ_1 dependent terms cancel each other, because $1-n_F(\beta\epsilon_k) = n_F(\beta\epsilon_k)e^{\beta\epsilon_k}$.
By knowing that at zero temperature we have $n_F(\beta\epsilon_k) = \theta(-\epsilon_k)$ and using the identity that $1-2\theta(-\epsilon_k) = \mathrm{sgn}(\epsilon_k)$ we have

$$I(\beta,\epsilon_k) = \frac{1-2\theta(-\epsilon_k)}{\epsilon_k} = \frac{\mathrm{sgn}(\epsilon_k)}{\epsilon_k} = \frac{1}{|\epsilon_k|}$$

c) The integral that we needed to evaluate is the following integral

$$\delta = \int_{D'}^{D}\frac{d\epsilon}{\epsilon}\frac{e^{\beta\epsilon}-1}{e^{\beta\epsilon}+1} =$$
$$\int_{\beta D'}^{\beta D}\frac{dx}{x}\frac{e^x-1}{e^x+1} \quad (5)$$

As we know $\beta D \gg 1$, now if $\beta D' \gg 1$ then in the whole range of integration $e^x \gg 1$ and we have $\frac{e^x-1}{e^x+1} \approx 1$ thus we get

$$\delta \approx \int_{\beta D'}^{\beta D}\frac{dx}{x} = \ln\frac{D}{D'} \quad (6)$$

which is the same result as before.
For the case of $\beta D' \ll 1$ we have

$$\delta = \int_{\beta D'}^{\beta D}\frac{dx}{x}\frac{e^x-1}{e^x+1} = \int_0^{\beta D}\frac{dx}{x}\frac{e^x-1}{e^x+1} - \int_0^{\beta D'}\frac{dx}{x}\frac{e^x-1}{e^x+1} \quad (7)$$

The dependence on D' solely comes from the last expression. As we assumed that $\beta D' \ll 1$, thus we could Taylor expand the exponetials of the last integral and we get

$$\delta \approx \int_0^{\beta D}\frac{dx}{x}\frac{e^x-1}{e^x+1} - \int_0^{\beta D'}\frac{dx}{x}\frac{x}{2}$$
$$= -\frac{\beta D'}{2} + \int_0^{\beta D}\frac{dx}{x}\frac{e^x-1}{e^x+1} \quad (8)$$

Which we see, weakly depends on D' and is not diverging in contrast to zero-temperature result.
d) Now suppose that we start the RG from a high-energy cut-off D_0 with coupling λ_0 and want to find the coupling at temperature T, by using the result of last part we get

$$\lambda(T) = \frac{\lambda_0}{1 - \lambda_0 \ln \frac{D_0}{T}} \tag{9}$$

Now let us use this relation for different temperature, let's say T_k where $\lambda(T_k) \approx 1$ thus we have

$$\lambda(T_k) = \frac{\lambda_0}{1 - \lambda_0 \ln \frac{D_0}{T_k}} \approx 1 \tag{10}$$

By solving the above equation we could find λ_0 and we get $\lambda_0 \approx 1/(1 + \ln \frac{D_0}{T_k})$. By using this relation in Eq. 9 we have

$$\lambda(T) \approx \frac{1}{1 + \ln \frac{D_0}{T_k} - \ln \frac{D_0}{T}} = \frac{1}{1 + \ln \frac{T}{T_k}} \approx \left(\ln \frac{T}{T_k}\right)^{-1} \tag{11}$$

2) We have

$$\frac{d\lambda}{-\lambda^2 + \lambda^3/2} = d \ln D \tag{12}$$

By integrating above equation we get

$$\left[\frac{1}{\lambda} + \frac{1}{2} \ln \frac{2-\lambda}{\lambda}\right]_{\lambda_0}^{\lambda_K} = \ln \frac{T_k}{E_F} \tag{13}$$

where λ_0 is the coupling at energy E_k. By using the fact that the Kondo coupling is approximately 1, $\lambda_K \approx 1$ and for $\lambda_0 \ll 1$ we get

$$\left[\frac{1}{\lambda} + \frac{1}{2} \ln \frac{2-\lambda}{\lambda}\right]_{\lambda_0}^{\lambda_K} = -\frac{1}{\lambda_0} - \frac{1}{2} \ln \frac{2-\lambda_0}{\lambda_0} + 1 \approx -\frac{1}{\lambda_0} - \frac{1}{2} \ln \frac{2}{\lambda_0} + 1 = \ln \left(e\sqrt{\frac{\lambda_0}{2}} e^{-1/\lambda_0}\right) \tag{14}$$

By using above equations we get

$$T_k \approx c * E_F \sqrt{\lambda_0} e^{-1/\lambda_0} \tag{15}$$

where c is a constant of order 1 not determined accurately by this procedure. Thus we get $p = \frac{1}{2}$. This leads to a slight reduction of T_K but it is relatively unimportant compared to the exponential factor. So, we see that we get a quite accurate result from knowing only the quadratic term in the β-function.