# Physics 503, Problem Set 3 Solutions

1a) By Taylor expansion we have

$$I(M) \equiv \int \sum_m \frac{1}{m!} \left[\prod_{i=1}^{2n} d\chi_i\right] \left[-\sum_{i,j} \chi_i M_{ij} \chi_j\right]^m \tag{1}$$

terms with $m < n$ are zero, because there is at least one $\chi_i$ which is not appear in the expansion and by using $\int d\chi = 0$ we get zero for any $m < n$. For all the terms with $m > n$ there exist at least one $\chi_i$ which appears twice in the expansion and as the $\chi_i$ are Grassmann numbers we have $\chi_i^2 = 0$. Thus it remains only $m = n$ term

$$\begin{aligned}
I(M) &\equiv \int \frac{1}{n!} \left[\prod_{i=1}^{2n} d\chi_i\right] \left[-\sum_{i,j} \chi_i M_{ij} \chi_j\right]^n \\
&= \int \frac{1}{n!} \sum_{\{i_1,i_2,\ldots i_{2n}\}} M_{i_1 i_2} M_{i_3 i_4} M_{i_5 i_6} \ldots M_{i_{2n-1} i_{2n}} \left[\prod_{i=1}^{2n} d\chi_i\right] \chi_{i_1}\chi_{i_2}\chi_{i_3}\chi_{i_4} \cdots \chi_{i_{2n-1}}\chi_{i_{2n}} \\
&= \frac{1}{n!} \sum_{\{i_1,i_2,\ldots i_{2n}\}} M_{i_1 i_2} M_{i_3 i_4} M_{i_5 i_6} \ldots M_{i_{2n-1} i_{2n}} \epsilon_{i_1 i_2 i_3 \ldots i_{2n}} \\
&= 2^n \mathrm{Pf}(M)
\end{aligned} \tag{2}$$

(In the second last line, above, the sum is over all permutations of the indices $\{1, 2, 3, \ldots n\}$.) The proportionality constant is $2^n$.

2b) By using above relation we have

$$\begin{aligned}
[\mathrm{Pf}(M)]^2 &= \textstyle\int 2^{-2n} \left[\prod_{i=1}^{2n} d\chi_i\right] \exp\left[-\sum_{i,j} \chi_i M_{ij} \chi_j\right] \left[\prod_{i=1}^{2n} d\eta_i\right] \exp\left[-\sum_{i,j} \eta_i M_{ij} \eta_j\right] \\
&= \int 2^{-2n} \left[\prod_{i=1}^{2n} d\chi_i\right] \left[\prod_{i=1}^{2n} d\eta_i\right] \exp\left[-\sum_{i,j} \chi_i M_{ij} \chi_j + \eta_i M_{ij} \eta_j\right]
\end{aligned} \tag{3}$$

By doing the change of variables from $\chi_i$ and $\eta_i$ to $\psi_j \equiv \chi_j + i\eta_j$ and $\bar{\psi}_j \equiv \chi_j - i\eta_j$ we have

$$\sum_{i,j} \chi_i M_{ij} \chi_j + \eta_i M_{ij} \eta_j = \sum_{i,j} \bar{\psi}_i M_{ij} \psi_j$$

and from Eq. 3 we get

$$[\mathrm{Pf}(M)]^2 = \int 2^{-2n} \left[\prod_{i=1}^{2n} d\bar{\psi}_i d\psi_i \mathrm{Det}[J_i]\right] \exp\left[-\sum_{i,j} \bar{\psi}_i M_{ij} \psi_j\right] \tag{4}$$

Where $\mathrm{Det}[J_i]$ is the determinant of the Jacobian of the change of variables. This can be evaluated by noting that:

$$\int d\chi d\eta \bar{\psi}\psi = \int d\chi d\eta (\chi - i\eta)(\chi + i\eta) = -2i = 2i \int d\bar{\psi} d\psi \bar{\psi}\psi. \tag{5}$$

Thus $\mathrm{Det}[J_j] = 2i$ for any $j$. Finally we get

$$[\mathrm{Pf}(M)]^2 = \int \left[\prod_{i=1}^{2n} d\bar{\psi}_i d\psi_i\right] \exp\left[-\sum_{i,j} \bar{\psi}_i M_{ij} \psi_j\right] = \mathrm{Det}(M)$$

2a)

$$Z = 1 + e^{-\beta(-t-\mu)} + e^{-\beta(t-\mu)} + e^{-\beta(V-2\mu)} \approx 1 + e^{-\beta(-t-\mu)} + e^{-\beta(t-\mu)} + e^{2\beta\mu} - e^{2\beta\mu}\beta V. \tag{6}$$

b) We may rewrite the Hamiltonian as:

$$\hat{H} - \mu\hat{N} = (-t-\mu)\hat{\psi}_+^\dagger \hat{\psi}_+ + (t-\mu)\hat{\psi}_-^\dagger \hat{\psi}_- + V\hat{\psi}_+^\dagger \hat{\psi}_-^\dagger \hat{\psi}_- \hat{\psi}_+. \tag{7}$$

Now consider the Feynman path integral. Expanding the Action to first order in $V$ we obtain:

$$\delta Z/Z = -V \int_0^\beta d\tau \langle \bar{\psi}_+(\tau)\bar{\psi}_-(\tau)\psi_-(\tau)\psi_+(\tau)\rangle \tag{8}$$

Here $\langle \ldots \rangle$ denotes the Grassmann integral weighted by the non-interacting action:

$$S_0 = \int_0^\beta d\tau[\bar{\psi}_+(\tau)(\tau)(-d/d\tau + t + \mu)\psi_+ + \bar{\psi}_-(\tau)(\tau)(-d/d\tau - t + \mu)\psi_-]. \tag{9}$$

As we see in the action $S_0$, $\psi_+$ and $\psi_-$ are decoupled, thus we have

$$\langle \bar{\psi}_+(\tau)\bar{\psi}_-(\tau)\psi_-(\tau)\psi_+(\tau)\rangle = \langle \bar{\psi}_+(\tau)\psi_+(\tau)\rangle\langle \bar{\psi}_-(\tau)\psi_-(\tau)\rangle \tag{10}$$

and from the Hamiltonian and using the hint of the problem we have $\langle \bar{\psi}_+(\tau)\psi_+(\tau)\rangle = n_F(-t-\mu)$ and $\langle \bar{\psi}_-(\tau)\psi_-(\tau)\rangle = n_F(t-\mu)$, thus we get

$$\delta Z/Z = -V\beta n_F(-t-\mu)n_F(t-\mu) = -\frac{V\beta}{1+e^{\beta(-t-\mu)}+e^{\beta(t-\mu)}+e^{-2\beta\mu}}. \tag{11}$$

From part a) we obtain:

$$\delta Z/Z = -\frac{\beta V e^{2\beta\mu}}{1+e^{-\beta(-t-\mu)}+e^{-\beta(t-\mu)}+e^{2\beta\mu}}. \tag{12}$$

Dividing through a factor of $e^{2\beta\mu}$ we see that this is the same result as obtained by the path integral perturbation theory.

3) Consider first a chain of just 4 sites, 0,1,2,3 with sites 1 and 2 strongly coupled with coupling $J$, and weak couplings $J'$ between 0-1 and 2-3. If we ignore the weak coupling $J'$ altogether, then the ground state is 4-fold degenerate: The spins on sites 1 and 2 form a singlet but the spins on sites 0 and 3 may be up or down. These four degenerate states correspond to a total spin singlet and triplet. The weak couplings $J'$ break this degeneracy and we may calculate the effect in second order degenerate perturbation theory. According to the standard result, the low energy eigenvalues and eigenvectors are determined by the effective Hamiltonian:

$$\langle i|H_{\text{eff}}|j\rangle = \sum_k{}' \langle i|H_{\text{int}}|k\rangle \frac{1}{E_0 - E_k}\langle k|H_{\text{int}}|j\rangle. \tag{13}$$

Here $|i\rangle$, $|j\rangle$ are the degenerate ground states and the sum over intermediate states $|k\rangle$ is restricted to excited states of the unperturbed Hamiltonian; $E_0$ and $E_k$ are the eigenstates of the unperturbed Hamiltonian. We may write an operator equation for the effective Hamiltonian:

$$H_{\text{eff}} = H_{\text{int}}\frac{1-P}{E_0 - H_0}H_{\text{int}} \tag{14}$$

where $P$ projects out the ground states of the unperturbed Hamiltonian $H_0$. In this case, acting on one of the 4 ground states,

$$H_{\text{int}} \equiv J'(\vec{S}_0 \cdot \vec{S}_1 + \vec{S}_2 \cdot \vec{S}_3) \tag{15}$$

always raises the state of the 2 spins on sites 1 and 2 to the excited triplet state. (So the projection operator $P$ is not necessary in this case.) Thus the energy difference is always $J$, the difference between singlet and triplet energies for the Hamiltonian $H_0 = J\vec{S}_1 \cdot \vec{S}_2 = J(1/2)[(\vec{S}_1 + \vec{S}_2)^2 - 3/2]$. We may write:

$$H_{\text{eff}} = -\frac{2J'^2}{J}\sum_{ab} S_0^a S_3^b \langle 0|S_1^a S_2^b|0\rangle. \tag{16}$$

Here $|0\rangle$ denotes the singlet state for the spins on sites 1 and 2. I have dropped the terms $\propto (S_0^a S_0^b + S_3^a S_3^b)$ which will simply shift $H_{\text{eff}}$ by a constant ("c-number"). The factor of 2 arises in Eq. (16) because there are two terms where either the first term in $H_{\rm int}$ occurs in the left hand factor in Eq. (13) and the second term in $H_{\rm int}$ occurs in the right hand factor, or vice versa. Acting on the singlet state:

$$S_1^a S_2^b|0\rangle = (\delta^{ab}/3)\vec{S}_1 \cdot \vec{S}_2|0\rangle = (\delta^{ab}/6)[(\vec{S}_1+\vec{S}_2)^2 - 3/2]|0\rangle = -(\delta^{ab}/4)|0\rangle. \tag{17}$$

Thus:

$$H_{\rm eff} = \frac{J'^2}{2J}\vec{S}_0 \cdot \vec{S}_3 + \text{constant}. \tag{18}$$

Projecting sites 1 and 2 onto their singlet ground state induces an antiferromagnetic exchange of magnitude $J_{\rm eff} = J'^2/(2J)$ between the 2 surrounding sites 0 and 3. This is true for every strongly coupled pair of spins so the low energy effective Hamiltonian is an antiferromagnetic Heisenberg Hamiltonian for the weakly coupled spins on sites 3i:

$$H_{\rm eff} = \frac{J'^2}{2J}\sum_i \vec{S}_{3i} \cdot \vec{S}_{3i+3} + \text{constant}. \tag{19}$$

Note that this simple projection method can be regarded as a sort of renormalization group procedure. We effectively integrated out the strongly coupled spins to obtain an effective Hamiltonian describing the low energy states which only contains the low energy degrees of freedom: the weakly coupled spins.