Physics 503, Problem Set 3, Due by Thursday, Feb. 23
(This Problem Set is due during the midterm break, a week after the midterm exam. Please bring it to my office then, or send a scan by email. I recommend at least trying all problems before the exam for practice.)

1) The Pfaffian of an anti-symmetric $2n \times 2n$ dimensional matrix, M is defined as:

$$\text{Pf}(M) \equiv \frac{1}{2^n n!} \sum_{\{i_1, i_2, \ldots i_n\}} \epsilon_{i_1 i_2 i_3 \ldots i_{2n}} M_{i_1 i_2} M_{i_3 i_4} M_{i5 i_6} \ldots M_{i_{2n-1} i_{2n}} \tag{1}$$

where $\epsilon_{i_1 i_2 i_3 \ldots i_{2n}}$ is the anti-symmetric 2n^{th}-rank unit tensor and the sum is over all permutations of the indices. The Pfaffian can be proven to obey the Pfaffian identity $[\text{Pf}(M)]^2$=Det (M); its square is the determinant. A short, elegant proof of this Pfaffian identity can be constructed using the Grassmann integral. Construct this proof as follows.
a) Consider the Grassmann integral:

$$I(M) \equiv \left[\prod_{i=1}^{2n} d\chi_i\right] \exp\left[-\sum_{i,j} \chi_i M_{ij} \chi_j\right] \tag{2}$$

Note we have $2n$ χ_j variables but no $\bar{\chi}_j$ variables. Show that $I(M) \propto \text{Pf}(M)$ and find the constant of proportionality.
b) Now consider $I(M)^2$. This can be written as a product of Grassmann integrals over 2 sets of Grassmann variables χ_j and η_j. Change Grassmann integration variables to:

$$\begin{aligned} \psi_j &\equiv \chi_j + i\eta_j \\ \bar{\psi}_j &\equiv \chi_j - i\eta_j \end{aligned} \tag{3}$$

being careful to calculate the Jacobean for the change of variables. Show that the resulting Grassmann integral is proportional to

$$\left[\prod_j d\bar{\psi}_j d\psi_j\right] \exp\left[-\sum_{i,j} \bar{\psi}_i M_{ij} \psi_j\right] = \text{Det}(M) \tag{4}$$

and thus complete the proof. [You may use Eq. (4) which was essentially proven in class.]

Grassmann integrals of the form of Eq. (2) occur for Majorana fermions which are used to describe certain quantum Hall states and other phenomena in condensed matter physics. Indeed a class of trial wave-functions for the fractional hall effect are referred to as Pfaffian states.

2) Consider again the simple 2-site tight-binding model for spinless electrons, that we studied in PS1 problem 1a) with:

$$\hat{H} - \mu\hat{N} = -t(\hat{\psi}_1^\dagger \hat{\psi}_2 + h.c.) + V\hat{\psi}_2^\dagger \hat{\psi}_1^\dagger \hat{\psi}_1 \hat{\psi}_2 - \mu(\hat{\psi}_1^\dagger \hat{\psi}_1 + \hat{\psi}_2^\dagger \hat{\psi}_2) \tag{5}$$

Here I have written the Hamiltonian in Shankar notation and I have included the chemical potential term.
a) Using our results from PS1, write the exact partition function for this model. Expand it to first order in V.
b) Rewrite the Hamiltonian of Eq. (5) in terms of the operators which diagonalize the hopping term:

$$\hat{\psi}_\pm \equiv (\hat{\psi}_1 \pm \hat{\psi}_2)/\sqrt{2}. \tag{6}$$

Write the Feynman path integral for the partition function of this model, introducing Grassmann variables ψ_+, ψ_-, $\bar{\psi}_+$ and $\bar{\psi}_-$. Expand the path integral to first order in V:

$$Z = Z_0 + VZ_1 + O(V^2). \tag{7}$$

Explicitly evaluate this expression. (Don't worry about the value of Z_0 itself, which may not be given correctly by the path integral; just focus on this ratio.)

You will find that you need to evaluate an ambiguous quantity, $< \bar{\psi}(\tau)\psi(\tau) >$ in the path integral. Since we concluded that $< \bar{\psi}(\tau)\psi(\tau') >$ equals the corresponding time-ordered Green's function in the operator formalism, it is unclear what value it has for equal times. Actually, a careful look at the derivation of the path integral resolves this ambiguity. In that derivation, we see that $\bar{\psi}$ occurs in the action at a time later by β/N than ψ where N is the number of time steps in the discretization of the path integral, assuming the Hamiltonian is written in normal ordered form as in Eq. (5). Therefore, we should interpret $<\bar{\psi}(\tau)\psi(\tau)>$ as $<\bar{\psi}(\tau+0^+)\psi(\tau)>$ where 0^+ is a positive infinitesimal quantity. Thus we obtain:

$$<\bar{\psi}(\tau+0^+)\psi(\tau)>=n_F(\epsilon) \tag{8}$$

where ϵ is the energy of the corresponding state.

Compare to the Taylor expansion of the exact answer, from part a).

3) Consider the quantum S=1/2 Heisenberg antiferromagnetic chain with non-uniform Hamiltonian:

$$H=\sum_i J_i \vec{S}_i \cdot \vec{S}_{i+1} \tag{9}$$

with $\vec{S}_i$ spin-1/2 operators, $\vec{S}_i=(1/2)\vec{\sigma}_i$. Assume that the (posative) exchange couplings vary with period 3, with $J_{3i+1}\equiv J$, $J_{3i-1}=J_{3i}\equiv J'$ and $0<J'\ll J$. We may derive a low energy Hamiltonian (for energies $\ll J$) by integrating out the strongly coupled spins $\vec{S}_{3i\pm 1}$ to obtain the effective Hamiltonian for the weakly coupled spins $\vec{S}_{3i}$. Find this low energy Hamiltonian to order J'^2/J.

HINT: It is not necessary or advisable to use any fancy path integral methods here. Just use a projection technique based on degenerate perturbation theory.