Physics 503 Midterm Exam - Solutions

1) Since the model is non-interacting, we just need to solve the Schroedinger equation for single particle states, corresponding to the matrix:

$$h = \begin{pmatrix} \Delta & -t \\ -t & -\Delta \end{pmatrix}. \tag{1}$$

The single particle eigenvalues are

$$\epsilon_\pm = \pm\sqrt{t^2 + \Delta^2}. \tag{2}$$

We can put at most one (spin-up) electron in each single particle state. The total energy of each state is the sum of the single particle energies of the electrons in it. So the complete spectrum, labelled by number of electrons, is:

n	E
0	0
1	$-\sqrt{t^2+\Delta^2}$
1	$\sqrt{t^2+\Delta^2}$
2	0

(3)

2) Taylor expanding and using:

$$\int d\bar\psi_1 d\psi_1 d\bar\psi_2 d\psi_2 \bar\psi_1\psi_1\bar\psi_2\psi_2 = 1, \tag{4}$$

we obtain

$$\int d\bar\psi_1 d\psi_1 d\bar\psi_2 d\psi_2 \exp[-\epsilon_1\bar\psi_1\psi_1 - \epsilon_2\bar\psi_2\psi_2 - \lambda\bar\psi_1\psi_1\bar\psi_2\psi_2] = \epsilon_1\epsilon_1 - \lambda. \tag{5}$$

3a) To see whether λ increases or decreases in magnitude we should check the sign of $d\lambda/\lambda$. Noting that $d\ln D < 0$ as we lower the cut-off energy, we see that $d\lambda/\lambda$ is positive for $c = -1$ and negative for $c = 1$, regardless of the sign of λ_0. Therefore λ increases in magnitude for $c = -1$, cases ii) and iv) but decreases in magnitude for $c = 1$, cases i) and iii). For $c = 1$, cases i) and iii), λ keeps on decreasing as we lower the cut off scale, D, so we don't need to know the high order terms in the β-function to conclude that it will renormalize to zero. On the other hand, for $c = -1$, cases ii) and iv), λ increases as we lower the cut off scale, eventually becoming O(1) so that higher order terms in the β-function will eventually become important.
b) The value of the renormalized coupling, λ, at a reduced cut off scale, D is given by

$$\int_{\lambda_0}^{\lambda} \frac{d\lambda}{\lambda^3} \approx c\ln(D/D_0) \tag{6}$$

giving:

$$\frac{1}{\lambda_0^2} - \frac{1}{\lambda^2} \approx 2c\ln(D/D_0). \tag{7}$$

Since $D < D_0$, this implies that if $c = 1$ the right hand side of Eq. (7) is negative and therefore $|\lambda| < |\lambda_0|$ and λ goes to zero as:

$$\lambda \to \frac{\mathrm{sign}(\lambda_0)}{[2\ln(D_0/D)]^{1/2}}. \tag{8}$$

On the other hand, if $c = -1$, $|\lambda|$ grows as D is reduced and equals one at a value of D given approximately by:

$$\frac{1}{\lambda_0^2} \approx 2\ln(D_0/D) + 1 \approx 2\ln(D_0/D). \tag{9}$$

Identifying this value of D with T_K and solving gives:

$$T_K \propto D_0 \exp[-1/(2\lambda_0^2)]. \tag{10}$$

[Keeping the 1 on the right hand side of Eq. (9) only changes the prefactor in Eq. (10) by a factor of order 1.] REMARK: In this case the characteristic energy scale, as a function of the bare coupling, is even smaller than in the Kondo model with its quadratic β-function.

4a) Noting that acting with ψ on any states lowers its energy by ϵ, we may write:

$$\psi(t) = e^{-i\epsilon t}\psi. \tag{11}$$

In thermal equilibrium:

$$\begin{aligned} &< \psi^\dagger \psi > = n_F(\beta\epsilon) \\ &< \psi \psi^\dagger > = 1 - n_F(\beta\epsilon). \end{aligned} \tag{12}$$

(The first line above follows since $\psi^\dagger\psi$ is the number operator. The second line follows using $\psi\psi^\dagger = 1 - \psi^\dagger\psi$.) Therefore,

$$\begin{aligned} G_T(t) &= -ie^{-i\epsilon t}[1 - n_F], \quad (t > 0) \\ &= ie^{-i\epsilon t} n_F, \quad (t < 0). \end{aligned} \tag{13}$$

This can also be written:

$$G_T(t) = ie^{-i\epsilon t}[n_F - \theta(t)]. \tag{14}$$

b)

$$\begin{aligned} G_T(\omega) &= in_F \int_{-\infty}^0 dt e^{[i(\omega-\epsilon)+\eta]t} - i[1 - n_F] \int_0^\infty dt e^{[i(\omega-\epsilon)-\eta]t} \\ &= \frac{n_F}{\omega - \epsilon - i\eta} + \frac{1 - n_F}{\omega - \epsilon + i\eta}. \end{aligned} \tag{15}$$

This can also be written:

$$G_T(\omega) = 2\pi i n_F \delta(\omega - \epsilon) + \frac{1}{\omega - \epsilon + i\eta}. \tag{16}$$

c)

$$\mathrm{Im}[G_T(\omega)] = [n_F - (1 - n_F)]\pi\delta(\omega - \epsilon). \tag{17}$$

Thus:

$$g(\beta\epsilon) = 1 - 2n_F(\beta\epsilon) = \tanh(\beta\epsilon/2). \tag{18}$$

REMARK: It can easily be proven in general for any Hamiltonian including interaction effects, using the spectral decomposition, that:

$$\begin{aligned} \mathrm{Im}[G_T(\omega)] &= \tanh(\beta\omega/2)\mathrm{Im}[G_R(\omega)] \\ \mathrm{Re}[G_T(\omega)] &= \mathrm{Re}[G_R(\omega)]. \end{aligned} \tag{19}$$

See, for example, Appendix 2 of Doniach and Sondheimer, *Green's Functions for Solid State Physicists.* At zero temperature these reduce to:

$$\begin{aligned} \mathrm{Im}[G_T(\omega)] &= \mathrm{sign}(\omega)\mathrm{Im}[G_R(\omega)] \\ \mathrm{Re}[G_T(\omega)] &= \mathrm{Re}[G_R(\omega)]. \end{aligned} \tag{20}$$

It is interesting to note that G_T is the analytic continuatiion of $\mathcal{G}$, the imaginary time Green's function, in time domain but not in frequency domain. On the other hand, $G_R(\omega)$ *is* the analytic continuation of $\mathcal{G}(i\omega_n)$ as discussed in class. This is one of the reasons why the retarded Green's function is more widely used in condensed matter physics than the real-time time-ordered Green's function. One is easily obtained from the other, in general, using Eq. (19).