(81)

- suppose we have $\hat{\psi}_i, \hat{\psi}_i^+$ with
$\{\hat{\psi}_i, \hat{\psi}_j^+\} = \delta_{ij}$

we introduce a pair of Grassmann numbers $\psi_i, \bar{\psi}_i$ for each $\hat{\psi}_i, \hat{\psi}_i^+$ pair of operators

- these obey $\{\psi_i, \bar{\psi}_j\} = 0 = \{\psi_i, \psi_j\} = \{\bar{\psi}_i, \bar{\psi}_j\}$

coherent states: eg - for 2 fermion ops:
$|\psi_1, \psi_2\rangle \equiv (1 - \psi_1 \hat{\psi}_1^+)(1 - \psi_2 \hat{\psi}_2^+)|0\rangle$

NB: 1-2 order doesn't matter but note $\psi\hat{\psi}^+$ order

- integral table again follows from $\int d\psi_i = 0$, $\int \psi_i d\psi_i = 1$

- consider $\int \exp\left[-\sum_{i,j=1}^{N} \bar{\psi}_i M_{ij} \psi_j\right] \prod_{i=1}^{N} d\bar{\psi}_i d\psi_i$

Which terms in Taylor expansion give non-zero integral?
- we Taylor expand exponential: only N^{th} order term has non-zero integral

$\int \exp[-\bar{\psi} M \psi] (d\bar{\psi} d\psi)$
$= \frac{(-1)^N}{N!} \int (\bar{\psi} M \psi)^N [d\bar{\psi} d\psi]$

- Consider $N=2$ case

$$\sum_{i,j,k,l=1}^{2} \int \bar{\Psi}_i M_{ij} \Psi_j \bar{\Psi}_k M_{kl} \Psi_l \, d\bar{\Psi}_1 d\Psi_1 d\bar{\Psi}_2 d\Psi_2$$

- only 2 non-zero terms each occurring twice

$$2 M_{11} M_{22} \int \bar{\Psi}_1 \Psi_1 \bar{\Psi}_2 \Psi_2 \, d\bar{\Psi}_1 d\Psi_1 d\bar{\Psi}_2 d\Psi_2$$

$$+ 2 M_{12} M_{21} \int \bar{\Psi}_1 \Psi_2 \bar{\Psi}_2 \Psi_1 \, d\bar{\Psi}_1 d\Psi_1 d\bar{\Psi}_2 d\Psi_2$$

$$= 2 \left(M_{11} M_{22} - M_{12} M_{21} \right) = 2 \operatorname{Det} M$$

so $\int \exp[-\bar{\Psi} M \Psi] (d\bar{\Psi} d\Psi) = \operatorname{Det} M$

for $N=2$

- generalizes to arbitrary N using

$$\int \exp[-\bar{\Psi} M \Psi] (d\bar{\Psi} d\Psi)$$

$$= \frac{1}{N!} \sum_{\substack{\{i_1 \dots i_N\} \\ \{j_1 \dots j_N\}}} M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_N j_N} \epsilon_{i_1 \dots i_N} \epsilon_{j_1 \dots j_N}$$

where $\epsilon_{i_1 \dots i_N} = \pm 1$ for $\{i_1 \dots i_N\}$ a permutation of $\{1 \dots N\}$, equals sign of permutation

$$= \sum_{\{j_1 \dots j_N\}} \epsilon_{j_1 \dots j_N} M_{1 j_1} \cdots M_{N j_N} = \operatorname{Det}(M)$$

- this is opposite of ~~the~~ ordinary integral

$\int \exp[-Z^* M Z] (dZ^* dZ) = (\mathrm{Det}\ M)^{-1}$

as we see by making a unitary transf to diagonalize M [assume M Hermitean & positive definite] gives $\prod_{i=1}^{N} \frac{1}{\lambda_i} = \frac{1}{\mathrm{Det} M}$

~~3/4/ — have $a|z\rangle = z|z\rangle$, discuss Sigma non-L.I. models~~

now resolution of identity is

$I = \int |\bar{\psi}_1 \ldots \bar{\psi}_N\rangle\langle \psi_1 \ldots \psi_N| \, e^{-\sum \bar{\psi}_i \psi_i} \prod d\bar{\psi}_i d\psi_i$

- basically follows from factorization

$|\bar{\psi}_1 \ldots \bar{\psi}_N\rangle\langle \psi_1 \ldots \psi_N| = |\bar{\psi}_1\rangle\langle\psi_1| \otimes |\bar{\psi}_2\rangle\langle\psi_2| \ldots \otimes |\bar{\psi}_N\rangle\langle\psi_N|$

$e^{-\sum \bar{\psi}_i \psi_i} = \prod_i e^{-\bar{\psi}_i \psi_i}$

and using fact that a <u>pair</u> of Grassman numbers commutes with everything

Path Integral for Fermions

- consider calculating $Z = e^{-\beta H}$ (or a Matsubara Green's function)

$$Z = \int \langle -\bar\psi | e^{-\beta H} | \psi \rangle \, e^{-\bar\psi \psi} \, d\bar\psi \, d\psi$$

eg - consider single level Hamiltonian

$H = \varepsilon \hat\psi^{+} \hat\psi$ [don't call it ϵ because, following Shankar, I will use ϵ for time steps]

since $\hat\psi | \psi \rangle = \psi | \psi \rangle$

and $\langle -\bar\psi | \hat\psi^{+} = \langle -\bar\psi | (-\bar\psi)$

Can we simply replace $\hat\psi \to \psi$, $\hat\psi^{+} \to \bar\psi$ in H?

- no - problem is that when we Taylor expand $e^{-\beta H}$ we don't get all $\hat\psi^{+}$'s on left & $\hat\psi$'s on right

(problem gets more severe when we include more degrees of freedom in H)

- to solve this problem we write

$$e^{-\beta H} = \left(e^{-\varepsilon H}\right)^N, \quad \varepsilon = \frac{\beta}{N} \to 0$$

- (for any H)

then use $e^{-\varepsilon H} \doteq 1 - \varepsilon H$

i.e. $\lim_{N \to \infty} \left(1 - \frac{\beta}{N} H\right)^N = e^{-\beta H}$

$$Z = tr\, e^{-\beta H} = tr\, (1 - \varepsilon H)^{N-1}$$

$$= \int \langle -\bar\Psi | (1-\varepsilon H)\, I\, (1-\varepsilon H)\, I \cdots I\, (1-\varepsilon H) | \Psi\rangle$$
$$e^{-\bar\Psi\Psi}\, d\bar\Psi\, d\Psi$$

- I inserted $N-2$ times

$$Z = \langle -\bar\Psi | 1-\varepsilon H | \Psi_{N-1}\rangle \langle \bar\Psi_{N-1} | 1-\varepsilon H | \Psi_{N-2}\rangle$$
$$\cdots (1-\varepsilon H) | \Psi_2\rangle \langle \bar\Psi_2 | 1-\varepsilon H | \Psi\rangle$$
$$e^{-\bar\Psi\Psi - \sum_{i=2}^{N-1} \bar\Psi_i \Psi_i}\, d\bar\Psi\, d\Psi \prod_{i=2}^{N-1} d\bar\Psi_i\, d\Psi_i$$

now rename dummy integration variable $(\bar\Psi_{N}, \Psi) \to (\bar\Psi_1, \Psi_1)$

- also let $-\psi_1 \equiv \psi_N$, $-\bar\psi_N = \bar\psi_1$

- NB $\psi_N, \bar\psi_N$ are not integrated over

so $\langle -\bar\psi| \to \langle -\bar\psi_1| \equiv \langle \bar\psi_N|$

$$Z = \prod_{j=2}^{N} \langle \bar\psi_j | (1-\varepsilon H) | \psi_{j-1} \rangle \, e^{-\sum_{i=1}^{N-1} \bar\psi_i \psi_i} \prod_{i=1}^{N-1} d\bar\psi_i \, d\psi_i$$

NB — anti-periodic b.c.'s because of minus sign in trace formula

- now, provided H is written with all creation ops to left of all annihilation ops we may use $\langle \bar\psi_j | H | \psi_{j-1} \rangle$

$= H(\bar\psi_j, \psi_{j-1})$

- i.e. replace operator $\psi^\dagger$ by $\bar\psi_j$ and ψ by ψ_{j-1} in each fact.

- since $\varepsilon \to 0$ we may also use $1 - \varepsilon \bar\psi_j \psi_{j-1} \to e^{-\varepsilon \bar\psi_j \psi_{j-1}}$

(87)

$$Z \to \int \left[\prod_{j=1}^{N-1} d\bar{\Psi}_j d\Psi_j\right] \exp\left[-\varepsilon \sum_{j=2}^{N} H(\bar{\Psi}_j, \Psi_{j-1})\right.$$

$$\left. + \sum_{j=2}^{N} \bar{\Psi}_j \Psi_{j-1} - \sum_{j=1}^{N-1} \bar{\Psi}_j \Psi_j \right]$$

where I used $\langle \bar{\Psi}_j | \Psi_{j-1} \rangle = e^{\bar{\Psi}_j \Psi_{j-1}}$

using $\Psi_1 = -\Psi_N$, $\bar{\Psi}_1 = -\bar{\Psi}_N$

$$Z = \int (d\bar{\Psi} d\Psi) \exp\left\{-\sum_{j=2}^{N}\left[\bar{\Psi}_j(\Psi_j - \Psi_{j-1}) + \varepsilon H(\bar{\Psi}_j, \Psi_{j-1})\right]\right\}$$

we now write $\Psi_j = \Psi(\varepsilon \cdot j)$

and regard $\varepsilon \cdot j$ as imaginary time

$\tau_j = \varepsilon \cdot j$, $\tau_j : \varepsilon, 2\varepsilon, \ldots N\varepsilon = \beta$

– we may regard $\Psi(\tau_j)$ as "slowly varying" ~~any~~ and use that to simplify expression [also $\bar{\Psi}(\tau_j)$]

actually since Ψ_j are Grassmann numbers it is not clear what slowly varying means

- we will be Fourier transforming $\psi(\tau_j)$

$\to \psi(\omega_n)$, $\omega_n = \frac{2\pi}{\beta}(n+\frac{1}{2})$ due to

anti-periodic BC

eg $\sum_j \bar{\psi}_j(\psi_j - \psi_{j-1})$

$\approx \int d\tau \, \bar{\psi} \frac{d\psi}{d\tau}$

$$\psi(\omega_n) = \varepsilon \sum_j e^{i\omega_n j} \psi(\tau_j) \approx \int d\tau \, e^{i\omega_n \tau} \psi(\tau)$$

$$\psi(\tau_j) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau_j} \psi(\omega_n)$$

[similarly for $\bar{\psi}$]

$$\sum_j \bar{\psi}_j(\psi_j - \psi_{j-1}) = \frac{1}{\beta^2} \sum_{n,m} \sum_j \bar{\psi}(\omega_n)\psi(\omega_m)$$

$$\sum_j e^{i(\omega_n - \omega_m)\tau_j} \left[1 - e^{+i\omega_m \varepsilon}\right]$$

$$= \frac{1}{\beta\varepsilon} \sum_n \bar{\psi}(\omega_n)\psi(\omega_n)\left[1 - e^{i\omega_n\varepsilon}\right]$$

$$\approx -\frac{i}{\beta} \sum_n \omega_n \bar{\psi}(\omega_n)\psi(\omega_n)$$

- as long as we calculate GF at

frequencies $<< \frac{1}{\varepsilon}$ we can approximate

$$\frac{1-e^{i\omega_n \varepsilon}}{\varepsilon} = -i\omega_n$$ which is equivalent

to replacing $\Psi_j - \Psi_{j-1}$ by $\varepsilon \frac{d\Psi}{d\tau}$

and $\sum_j \to \frac{1}{\varepsilon}\int d\tau$

- Similarly, we may replace

$$\varepsilon \sum_{j=2}^{N} H(\bar{\Psi}_j, \Psi_{j-1})$$

$$\approx \varepsilon \sum_{j=2}^{N} H(\bar{\Psi}_j, \Psi_j)$$

$$= \int_0^\beta d\tau\, H[\bar{\Psi}(\tau), \Psi(\tau)]$$

$$Z = \int [d\bar{\Psi} d\Psi]\, e^{S_I}$$

where $[d\bar{\Psi} d\Psi]$ now means a path integral

and $S_I = -\int_0^\beta d\tau \left[\bar{\Psi}\frac{d\Psi}{d\tau} + H[\bar{\Psi},\Psi]\right]$

S_I is the continuation to imaginary time of the canonical action

$$S = \int dt \left[i \psi^\dagger \frac{d\psi}{dt} - H \right]$$

$$i\int dt \to \int d\tau$$

$$e^{iS} \to e^{S_I}$$

- this all carries over to Hamiltonians with more operators [as long as all $\hat{\psi}^\dagger$ factors are on right]

- now consider calculating ~~the~~ Matsubara GF using Feynman Path Integral

$$\mathcal{G}(\tau_1, \tau_2) = - T \langle \hat{\psi}(\tau_1)\, \hat{\psi}^\dagger(\tau_2) \rangle, \quad 0 \le \tau_1, \tau_2 \le \beta$$

- if $\tau_1 > \tau_2$ we get

~~$\langle \bar{\psi}_N | (1-\varepsilon H) | \psi_{N-1} \rangle \langle \bar{\psi}_{N-1} | \cdots \langle \bar{\psi}_j (1-\varepsilon$~~

$$\mathcal{G} = -\frac{1}{Z}\, tr\, e^{-\beta H}\, \hat{\psi}(\tau_1)\, \hat{\psi}^\dagger(\tau_2)$$

$$= -\frac{1}{Z}\, tr \left[e^{-(\beta-\tau_1)H}\, \hat{\psi}\, e^{-(\tau_1-\tau_2)H}\, \hat{\psi}^\dagger\, e^{-\tau_2 H} \right]$$

- we get $\langle \bar{\psi}_j | \hat{\psi}^\dagger (1-\varepsilon H) | \psi_{j-1} \rangle$

at time step $\tau_j = \tau_1$

(8)

and $\langle \bar{\psi}_{j+1} | (1-\varepsilon H)\,\psi\, | \psi_j \rangle$ at time

step $\tau_j = \tau_2 \;\to\; \psi(\tau_2)\,\bar{\psi}(\tau_1)$

factors inside path integral

$$\mathcal{G}(\tau_1 - \tau_2) = -\frac{1}{Z}\int e^{S}\,\psi(\tau_1)\,\bar{\psi}(\tau_2)\,[d\bar{\psi}\,d\psi]$$

Feb 2 $\qquad\qquad \tau_1 > \tau_2$

- if $\tau_2 > \tau_1$ then $\bar{\psi}(\tau)$ factor appears

to left of $\psi(\tau_2)$

$$\mathcal{G}(\tau_1-\tau_2) = \cancel{-\frac{1}{Z}\int e^{S}\,\bar{\psi}(\tau_2)\,\psi(\tau_1)}\,[d\bar{\psi}\,d\psi]$$

$$= \cancel{\frac{1}{Z}\int e^{S}\,\psi(\tau_1)\,\bar{\psi}(\tau_2)\,[d\bar{\psi}\,d\psi]}$$

$\cancel{\tau_2 > \tau_1}$

$$= +\frac{1}{Z}\,\mathrm{tr}\left(e^{-(\beta-\tau_2)H}\,\hat{\psi}^{+}\,e^{-(\tau_2-\tau_1)H}\,\hat{\psi}\,e^{-\tau_1 H}\right)$$

$$= +\int e^{S}\,\bar{\psi}(\tau_2)\,\psi(\tau_1)\,[d\bar{\psi}\,d\psi]$$

- Grassmann variables anti-commute

so $\mathcal{G}(\tau_1-\tau_2) = -\frac{1}{Z}\int e^{S}\,\psi(\tau_1)\,\bar{\psi}(\tau_2)\,[d\bar{\psi}\,d\psi]$

regardless of sign of $\tau_1 - \tau_2$

$$\mathcal{G}(\tau_1 - \tau_2) = - \langle \psi(\tau_1)\, \bar{\psi}(\tau_2) \rangle$$

where $\langle\ \rangle$ denotes "average"

weighted by e^{S} and integrated over $[d\bar{\psi}\, d\psi]$

- Feynman path integral is automatically time ordered

- this is one of reasons why $(-)$ sign is appropriate in def^n of t-ordering for fermions & why Matsubara GF is useful

- carries over to multi-point GF's

$$\langle \psi(\tau_1) \cdots \psi(\tau_n)\, \bar{\psi}(\tau_{n+1}) \cdots \bar{\psi}(\tau_{2n}) \rangle$$

$$= e^{\beta \Omega}\, \mathrm{tr}\, e^{-\beta H}\, T\left[\hat{\psi}(\tau_1) \cdots \hat{\psi}(\tau_n)\, \hat{\psi}^{+}(\tau_{n+1}) \cdots \hat{\psi}^{+}(\tau_{2n}) \right]$$

- let's check free electron GF

$$S = -\int \left[\bar{\psi} \frac{d\psi}{d\tau} + \xi\, \bar{\psi}\psi \right]$$

$$= \frac{1}{\beta} \sum_n \bar{\psi}(\omega_n)\, \psi(\omega_n)\, (i\omega_n - \xi)$$

by above def^n of Fourier transform

- Grassmann integrals factorise

$$-\langle \psi(\omega_n)\bar{\psi}(\omega_n)\rangle = \int \cancel{d\bar{\psi}(\omega_n)\, d\psi(\omega_n)}\, e$$

$$= \langle \psi\bar{\psi}\rangle = \frac{\int d\bar{\psi}\, d\psi \; e^{\bar{\psi}\psi\,(i\omega_n - \xi)/\beta}\; \bar{\psi}\psi}{\int d\bar{\psi}\, d\psi \; e^{\bar{\psi}\psi\,(i\omega_n-\xi)/\beta}}$$

$$= \frac{\beta}{i\omega_n - \xi}$$

note that $\mathcal{G}(i\omega_n) = -\int_0^\beta d\tau\, e^{i\omega_n \tau} \langle \psi(\tau)\bar{\psi}^*(0)\rangle$

$$= -\langle \psi(\omega_n)\,.\,\bar{\psi}(\tau=0)\rangle$$

$$= -\frac{1}{\beta}\langle \psi(\omega_n)\bar{\psi}(\omega_n)\rangle$$

$$= \frac{1}{i\omega_n - \xi}$$

- correct answer

- simply the inverse of quadratic of coefficient of $\bar{\psi}(\omega_n)\psi(\omega_n)$ in S $[\div \beta]$

- It extends to multi-fermions models and multi-point GF's

- we dodge perturbation theory in H_{INT} by Taylor expanding $e^{-\int_0^\beta d\tau\, H_{INT}[\bar{\Psi}(\tau), \Psi(\tau)]}$ factor in e^S

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_0^\beta H d\tau\right)^n$$

- must evaluate Feynman integrals like $\frac{1}{\beta^2} \langle \bar{\Psi}(\omega_1) \Psi(\omega_2) \bar{\Psi}(\omega_3) \Psi(\omega_4) \rangle$

$$= \frac{1}{\beta^2} \Big[\int \omega_1, \omega_2 \langle \bar{\Psi}(\omega_1) \Psi(\omega_2) \rangle \int \omega_3, \omega_4 \langle \bar{\Psi}(\omega_3) \Psi(\omega_4) \rangle$$

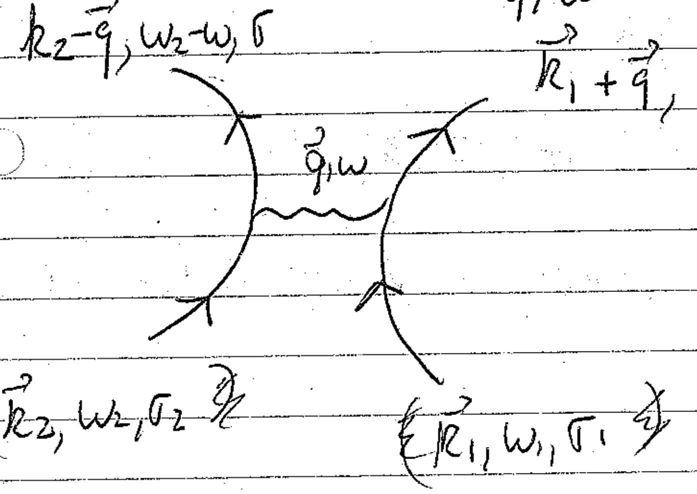
$$- \int \omega_1, \omega_4 \langle \bar{\Psi}(\omega_1) \Psi(\omega_4) \rangle \int \omega_2, \omega_3 \langle \bar{\Psi}(\omega_3) \Psi(\omega_2) \rangle \Big]$$

[at $\beta \to \infty$]

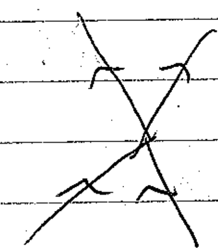
- note that minus sign due to Fermi statistics - otherwise has similar structure to bosonic case we can write jellium model, for example, ~~in~~ as a path integral $S_I = S_0 + S_{INT}$

$$S_0 = \frac{1}{\beta} \sum_{\omega_n, \vec{k}} \bar{\Psi}(\vec{k}, \omega_n)\, \Psi(\vec{k}, \omega_n)\, (i\omega_n - \epsilon_k)$$

$$S_{INT} = -\frac{1}{2V\beta^3} \sum_{\substack{(\vec{k}_1, \omega_1, \sigma_1) \\ (\vec{k}_2, \omega_2, \sigma_2) \\ \vec{q}, \omega}} V_{\vec{q}}\, \bar{\Psi}_{\vec{k}_1+\vec{q}, \omega_1+\omega, \sigma_1} \bar{\Psi}_{\vec{k}_2-\vec{q}, \omega_2-\omega, \sigma_2}\, \Psi_{\vec{k}_2, \omega_2, \sigma_2} \Psi_{\vec{k}_1, \omega_1, \sigma_1}$$



or simply



momentum, imaginary frequency and spin are conserved at vertex which has a factor of $V_{\vec{q}}$ associated with it

at $T \to 0$, $\Delta\omega_n = 2\pi T \to 0$ so $\frac{1}{\beta}\sum_{\omega_n} \to \int \frac{d\omega}{2\pi}$

factors of $\frac{1}{\beta}$ in S_I cancel

- we will use path integral to develop RG
- we will integrate out high energy states, far from Fermi surface — large $|\epsilon_{\vec{k}}|$ to get a low energy effective Hamiltonian (i.e. action) for states close to Fermi surface
- this gets complicated for $d=2$ or 3 due to presence of Fermi surface
- so will first discuss ~~$d=0$ and 1~~ simpler examples, that are still experimentally relevant

<u>The Kondo model</u>

- describes otherwise free conduction electrons interacting with a <u>single</u> magnetic impurity (at $\vec{r}=0$)
- appropriate model for dilute [eg $n \sim 10^{-6}$] magnetic impurities in a metal

- eg $Cu_{1-x}Fe_x$, $x \ll 1$
- also used to describe a quantum dot [produced by gating a semi-conductor hetero-structure] with an odd number of electrons [& hence a net spin] interacting with conduction electrons in heterostructure
- we hope we can ignore e-e interactions on basis of Fermi liquid theory
- there is also a F.L.T. for impurity interaction itself - however local F.L.T.
- but this only becomes applicable at low $T \ll T_K \ll E_F$ as we shall see
- ~~the Kondo~~ the Kondo model was developed and studied to explain a remarkable experimental observation

Going back several decades:

Some metals have a resistivity that increases as T is lowered, at low T!

- simplest version of the model

$$H = \sum_{\vec{k}} C^\dagger_{\vec{k}\sigma} C_{\vec{k}\sigma} \, \epsilon(\vec{k})$$

$$+ \frac{J}{V} \sum_{\substack{\vec{k},\vec{k}' \\ \alpha\beta}} C^\dagger_{\vec{k}\alpha} \frac{\vec{\sigma}_{\alpha\beta}}{2} C_{\vec{k}'\beta} \cdot \vec{S}$$

where $\vec{S} = \frac{1}{2}\vec{\sigma}$ are a vector of $S = \frac{1}{2}$
$[S^a, S^b] = i\,\epsilon^{abc} S^c$
operators acting on impurity spin (which is in up or down state)

$$H_{INT} = \frac{J}{2V} \sum_{\vec{k}\vec{k}'} \Big[\left(C^\dagger_{\vec{k}\uparrow} C_{\vec{k}'\downarrow} \, S^- + h.c. \right) + \left(C^\dagger_{\vec{k}\uparrow} C_{\vec{k}'\uparrow} - C^\dagger_{\vec{k}\downarrow} C_{\vec{k}'\downarrow} \right) S^z \Big]$$

- electron spin can flip during scattering process with impurity spin flipping to conserve total spin

- this apparently simple, yet quantum mechanical aspect of the Kondo interaction leads to all its complexities & richness
- otherwise we would have a simple potential scattering model which doesn't exhibit any interesting RG behavior
- in position space

$$H_{INT} = J \sum_{\alpha,\beta} \psi^{\dagger}_{\alpha}(\vec{r}=0) \; \frac{\vec{\sigma}_{\alpha\beta}}{2} \psi_{\beta}(\vec{r}=0) \cdot \vec{S}$$

- δ- function scattering (for simplicity)
- actually we need to cut-off interaction at large $\vec{k}, \vec{k}'$ because δ- function scattering potential is not well-defined in d=2 or 3
- i.e. use finite-range model.
- or can use a tight-binding model

$$H = -t \sum_{\vec{R}, \vec{\delta}, \sigma} C^{\dagger}_{\vec{R}\sigma} C_{\vec{R}+\vec{\delta}, \sigma} + J \sum_{\alpha\beta} C^{\dagger}_{0\alpha} \frac{\vec{\sigma}_{\alpha\beta}}{2} C_{0\beta} \cdot \vec{S}$$

- in continuum (k-space) model J has dimension of energy-volume
- relevant dimensionless parameter is $J\nu$ where ν is density of states at the Fermi energy - per unit energy, per unit volume, per spin component, in non-interacting theory
- for $\mathcal{E} = \frac{k^2}{2m}$

$$\nu = \int \frac{d^3k}{(2\pi)^3} \delta\left[\frac{k^2}{2m} - \mathcal{E}_F\right]$$

$$= \frac{mk_F}{2\pi^2} = \frac{k_F^3}{4\pi^2 \mathcal{E}_F}$$

- in general $\nu \sim \frac{1}{\mathcal{E}_F - a^3}$

where a = lattice spacing or e-e separation