and integrate over $\vec{S}_{\vec{r}}(t)$ with appropriate "Boltzmann factor" → action

- in some cases, quantum problem in d spatial dimensions is equivalent to classical problem in (d+1) spatial dimensions
- this is exactly true for Lorentz-invariant quantum field theories

not exactly true for any models in CMP but is approximately true in some cases [quantum spin models]

<u>Simple Example - Perturbation Theory for Classical XY model in d-dimensions</u>

- "easy plane" classical ferromagnet - spin vectors constrained to lie in plane

$$H = -J \sum_{\vec{R},\vec{\delta}} \vec{S}_{\vec{R}} \cdot \vec{S}_{\vec{R}+\vec{\delta}}$$

where $\vec{S}_{\vec{R}} = S(\cos \Phi_{\vec{R}}, \sin \Phi_{\vec{R}})$

$$\cancel{H = -J S^2 \sum_{\vec{R},\vec{\delta}} \cos}$$

let $\Phi_{\vec{R}} = S^x_{\vec{R}} + i S^y_{\vec{R}}$, $|\Phi_{\vec{R}}|^2 = S^2$

$$H = -\frac{J}{2T} \sum \left(\Phi_{\vec{R}}^* \Phi_{\vec{R}+\vec{\delta}} + cc \right)$$

$$= \frac{J}{2T} \sum \left| \Phi_{\vec{R}+\vec{\delta}} - \Phi_{\vec{R}} \right|^2 + \text{const}$$

$$Z = \prod_{\vec{R}} \int d \cancel{\Phi_{\vec{R}} d\Phi^*_{\vec{R}}}, \int d\varphi_{\vec{R}} \, e^{-H/T}$$

- N.B. - Cannot be done exactly - leads to phase transitions as function of T, critical phenomena

- some analogy between $\Phi_{\vec{R}}$ and $\Psi(\vec{r}, \tau)$ in Feynman path integral for interacting fermions - makes this worth studying in fact, it is more useful to study a "soft spin" version of XY model

- relax restraint $|\phi_{\vec{R}}|^2 = S^2$ and add $|\phi_{\vec{R}}|^2$, $|\phi_{\vec{R}}|^4$ terms to H

- can be justified by RG - same universality class

- soft-spin model has the advantage that we can simply Fourier transform

- for volume with N lattice sites

$$\phi_{\vec{R}} = \frac{1}{\sqrt{N}} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{R}} \phi(\vec{q})$$

$$\phi(\vec{q}) = \frac{1}{\sqrt{N}} \sum_{\vec{R}} e^{-i\vec{q}\cdot\vec{R}} \phi_{\vec{R}}$$

- quadratic terms in H are diagonal in momentum space:

$$\sum_{\vec{R},\delta} |\Phi_{\vec{R}+\vec{\delta}} - \Phi_{\vec{R}}|^2$$

$$= \sum_{\vec{q}} |\phi(\vec{q})|^2 \sum_{\vec{\delta}} |e^{i\vec{q}\cdot\vec{\delta}} - 1|^2$$

- consider simple cubic lattice in d-dimensions

$$\sum_{\vec{\delta}} |e^{i\vec{q}\cdot\vec{\delta}} - 1|^2 = 2 \sum_{j=1}^{d} |e^{i q_j a} - 1|^2$$

[a = lattice spacing]

$$= 8 \sum_j \sin^2 \frac{q_j a}{2} \simeq 2 \vec{q}^2 a^2 \text{ for } |\vec{q}| \ll \frac{1}{a}$$

- sums over $\vec{q}$ restricted to BZ: $|q_j| < \pi/a$
- for convenience we will restrict $\vec{q}$'s to $|\vec{q}| < \Lambda \ll \pi/a$.
- this is partly by way of illustration but also occurs in RG for classical XY model

$$\frac{H}{T} = \sum_{\vec{q}} (q^2 + m^2) |\phi(\vec{q})|^2 + \frac{u}{4N} \sum_{\vec{q}_1 - \vec{q}_4} \phi^{\alpha}(\vec{q}_4) \phi^{\alpha}(\vec{q}_3) \phi(\vec{q}_2) \phi(\vec{q}_1) \, \delta_{\vec{q}_1 + \vec{q}_2 - \vec{q}_3 - \vec{q}_4, \vec{0}}$$

- after rescaling of J, a, b etc.
- this is same as Shankar's (42), (72)
[in notation as in QFT]
except: - he takes infinite volume limit $N \to \infty$

$$\sum_{\vec{q}} \to \bar{V} \int \frac{d^3 \vec{q}}{(2\pi)^3}; \quad \cancel{Na^3} \quad \bar{V} = N a^3$$

- he also defines $\phi_{\vec{q}}$ to be larger by a factor of $\sqrt{\bar{V}}$ and sets lattice spacing $a = 1$

$$\sum_{\vec{q}} |\phi(\vec{q})|^2 \to \int \frac{d^3 q}{(2\pi)^3} |\phi(\vec{q})|^2$$

Kronecker δ-function gets replaced by Dirac δ-fn

$$\delta_{\vec{q},0} \to \frac{(2\pi)^d}{\bar{V}} \delta^d(\vec{q})$$

- quartic term

$$\to \frac{u (2\pi)^d}{4} \int \frac{d^d q_1 \cdots d^d q_4}{(2\pi)^{4d}} \phi^* \phi^* \phi \phi \, \delta^d [\vec{q}_1 + \vec{q}_2 - \vec{q}_3 - \vec{q}_4]$$

- ~~we want to calculate objects like~~

~~partition function~~

- Boltzmann sum is conveniently written in Fourier space

$$\prod_{\vec{R}} \int d\phi_{\vec{R}}\, d\phi^*_{\vec{R}} \propto \prod_{|\vec{q}|<\Lambda} \int d\phi(\vec{q})\, d\phi^*(\vec{q})$$

- we define $\int [d\phi(\vec{q})\, d\phi^*(\vec{q})] \equiv \prod_{|\vec{q}|<\Lambda} \frac{1}{\pi} \int_{-\infty}^{\infty} d\mathrm{Re}\,\phi(\vec{q})\, d\mathrm{Im}\,\phi(\vec{q})$

as in Shankar (44)

$$Z \propto \int [d\phi\, d\phi^*]\, e^{-H/T}$$

- we also want classical Green's function

$$\langle \phi^*(\vec{q}_1)\, \phi(\vec{q}_2) \rangle \equiv \frac{1}{Z} \int [d\phi^* d\phi]\, \phi^*(\vec{q}_1) \phi(\vec{q}_2)\, e^{-H/T}$$

- for $u=0$ we can evaluate these exactly, otherwise we do perturbation theory in u

- <u>Partition Function</u>

- factorizes into separate integral for each $\vec{q}$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 \; e^{-(\phi_1^2+\phi_2^2)(q^2+m^2)}$$

(where $\phi(\vec{q}) = \phi_1 + i\,\phi_2$)

$$= \frac{1}{q^2+m^2}$$

$$Z = \prod_{|\vec{q}| \geq \Lambda} \frac{1}{q^2+m^2}$$

clearly $\langle \phi^*(\vec{q}_1)\,\phi(\vec{q}_2) \rangle$ vanishes unless $\vec{q}_1 = \vec{q}_2$, also $\langle \phi\,\phi \rangle = \langle \phi^*\phi^* \rangle = 0$ (check!)

$$\langle |\phi(\vec{q})|^2 \rangle = (q^2+m^2) \int \frac{d\phi_1\, d\phi_2}{\pi} (\phi_1^2+\phi_2^2)\, e^{-(\phi_1^2+\phi_2^2)(q^2+m^2)}$$

– since integrals cancel in numerator and denominator for all other values of $\vec{q}$

$$\int_{-\infty}^{\infty} \frac{d\phi_1}{\sqrt{\pi}}\, e^{-\phi_1^2 (q^2+m^2)} = \frac{1}{\sqrt{q^2+m^2}}$$

$$\int_{-\infty}^{\infty} \frac{d\phi_1}{\sqrt{\pi}}\, \phi_1^2\, e^{-\phi_1^2 (q^2+m^2)} = -\frac{d}{dm^2}\frac{1}{\sqrt{q^2+m^2}} = \frac{1}{2(q^2+m^2)^{3/2}}$$

so $\langle |\phi(\vec{q})|^2 \rangle = \dfrac{1}{q^2+m^2}$

<u>2b)1</u> $\langle \phi^*(\vec{q}_1)\, \phi(\vec{q}_2) \rangle = 0$ if $\vec{q}_1 \neq \vec{q}_2$ why?

- as a preliminary to perturbation theory in u, consider

$$\langle \phi^*(\vec{q}_4)\, \phi^*(\vec{q}_3)\, \phi(\vec{q}_2)\, \phi(\vec{q}_1) \rangle \text{ at } u=0$$

this vanishes unless $(\vec{q}_1 = \vec{q}_3,\ \vec{q}_2 = \vec{q}_4)$

<u>or</u> $(\vec{q}_1 = \vec{q}_4,\ \vec{q}_2 = \vec{q}_3)$

- if $\vec{q}_1, \vec{q}_2$ are different, it then simply factorizes

$$= \langle \phi^*(\vec{q}_3)\, \phi(\vec{q}_1) \rangle \langle \phi^*(\vec{q}_4)\, \phi(\vec{q}_2) \rangle + \langle \phi^*(\vec{q}_4)\, \phi(\vec{q}_1) \rangle \langle \phi^*(\vec{q}_3)\, \phi(\vec{q}_2) \rangle$$

$$= \frac{\delta_{\vec{q}_1, \vec{q}_3}\, \delta_{\vec{q}_2, \vec{q}_4} + \delta_{\vec{q}_1, \vec{q}_4}\, \delta_{\vec{q}_2, \vec{q}_3}}{(q_1^2+m^2)(q_2^2+m^2)}$$

NB - we aren't interested in case $\vec{q}_1 = \vec{q}_2 = \vec{q}_3 = \vec{q}_4$, which requires special evaluation, because we will be

doing sums over $\vec{q}_i$ [integrals as $V \to \infty$]
and this is a point of measure zero
[giving contribution suppressed by $\frac{1}{V}$]

- now consider perturbation theory in u
for Z:

$$\frac{Z}{Z_0} = 1 - \frac{u}{4V} \sum_{\vec{q}_1 \cdots \vec{q}_4} \langle \phi^*(\vec{q}_4)\,\phi^*(\vec{q}_3)\,\phi(\vec{q}_2)\,\phi(\vec{q}_1) \rangle + \cdots$$

from expanding $e^{-\frac{u}{4} \cdots}$ to 1st order in u.

$\langle \cdots \rangle$ ~~denotes~~ denotes $\frac{1}{Z_0} \int [d\phi^* d\phi] \cdots$

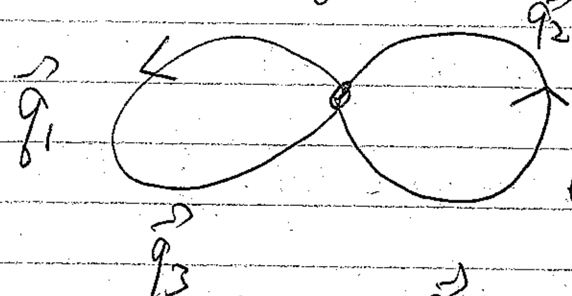
$$\frac{Z}{Z_0} = 1 - \frac{u}{2V} \left(\sum_{\vec{q}_1} \frac{1}{q_1^2 + m^2} \right)^2$$

$$\to 1 - \frac{u}{2} V \left(\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} \right)^2$$

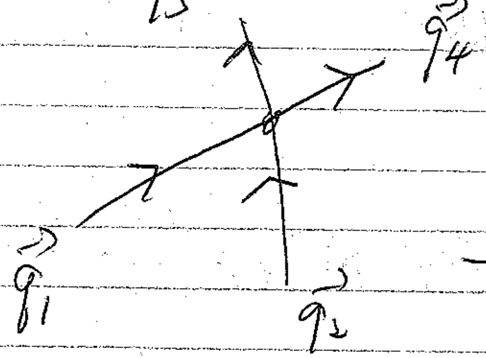
$$\simeq e^{-\frac{u}{2} V (\cdots)^2 + \cdots}$$

$$\frac{F}{T} \simeq \frac{F_0}{T} + \frac{u}{2} V (\cdots)^2$$ free energy extensive

- this is just 1^{st} term in ∞ series



- keep track of them with Feynman diagrams



represents interaction term

- inward arrow - ϕ
- outward arrow - ϕ^*

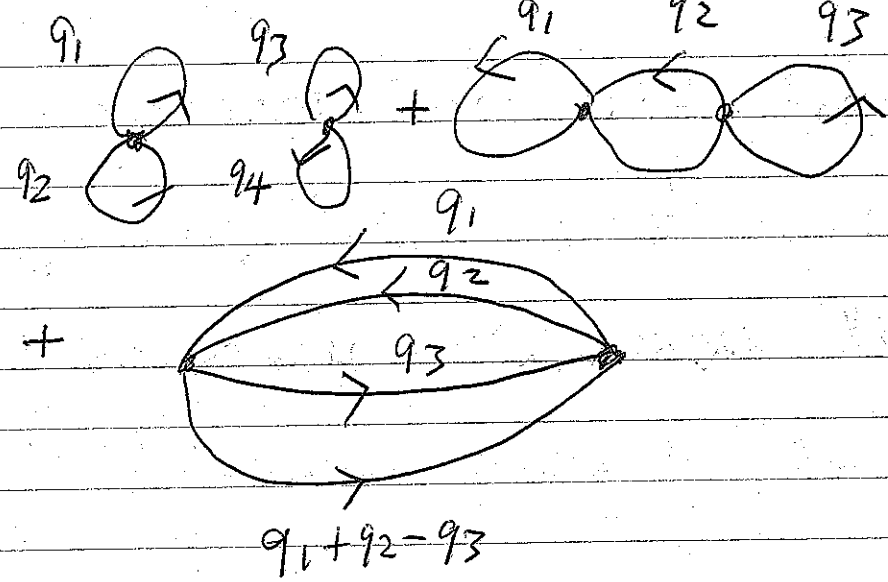
- we conserve momentum at each vertex

$$\propto \delta_{\vec{q}_1+\vec{q}_2-\vec{q}_3-\vec{q}_4,\,\vec{0}}$$

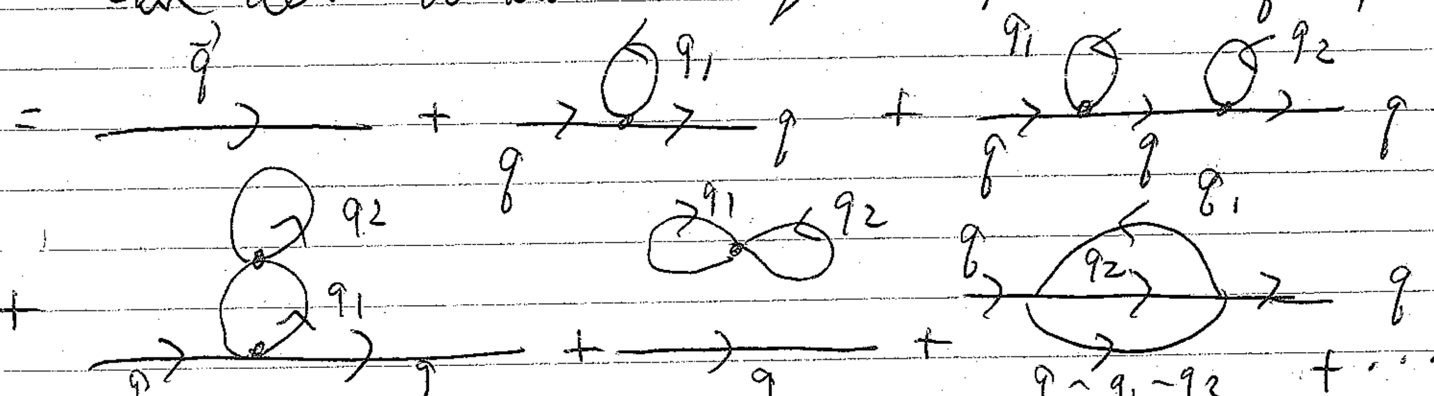
- we get a factor of $\frac{1}{q^2+m^2}$ for each line

- we integrate over momentum (wave-vector) on each ~~line~~ loop

2nd order



- each term gives $\frac{1}{q^2+m^2}$ factor for each line, $\int \frac{d^d q}{(2\pi)^d}$ for each independent wave-vector q_1, q_2, q_3 or q_1, q_2, q_3, q_4 (each loop)
- also some combinatoric factors + minus signs
- I will evaluate these when needed but not bother to develop systematic rules
- can also draw diagrams for $\langle \phi_q \phi_{q'} \rangle$



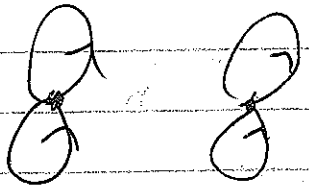
2 external lines with momentum $\vec{q}$
arise from $\phi(\vec{q})$, $\phi(\vec{q})^*$ factors
- internal lines arise from expanding in u
- this simple structure arises from
factorizing Gaussian integrals with
independent $\vec{q}_i$ for each pair
$\langle \phi(q_i) \phi^*(q_i) \rangle$

- summing over diagrams corresponds to
different pairings

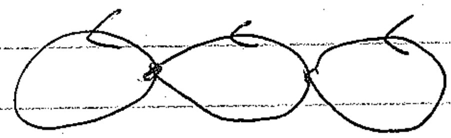
eg - consider $O(u^2)$ terms for Z

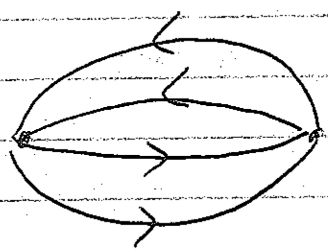
$$\propto u^2 \sum_{\vec{q}_1 - \vec{q}_8} \langle \phi^*(q_8)\phi^*(q_7)\phi(q_6)\phi(q_5)\phi^*(q_4)\phi^*(q_3)\phi(q_2)\phi(q_1)\rangle$$
$$\cdot\, \delta_{q_8+q_7,\, q_6+q_5}\, \delta_{q_4+q_3,\, q_2+q_1}$$

- we can ~~denote~~ denote $\phi^* \phi$ pairing by
a line $\sqcap$

 corresponds to

$\phi^* \phi^* \phi \phi \quad \phi^* \phi^* \phi \phi$

 to $\phi^* \phi^* \phi \phi \quad \phi^* \phi^* \phi \phi$

 to $\phi^* \phi^* \phi \phi \quad \phi^* \phi^* \phi \phi$

- these are only possibilities as can easily be seen.
- Feynman path integral for bosonic many body theory [QFT] has similar structure, similar Feynman diagrams, but I won't take the time to derive it
- Shankar discusses RG for classical XY model - I also won't take time

to discuss that

- instead we will forge ahead to fermionic Feynman path integrals & RG for Kondo model

Feynman Path Integral for Fermions

- both bosonic & fermionic path integrals are based on inserting complete sets of states, $I = \sum |n\rangle\langle n|$ many times

$$e^{iHt} = \left(e^{iH\Delta t}\right)^{N} \qquad (\Delta t = t/N)$$

$$= e^{iH\Delta t} \sum |n_1\rangle\langle n_1| \; e^{iH\Delta t} \sum |n_2\rangle\langle n_2| \; \cdots$$

- we need to find a convenient representation for this

- consider first a single harmonic oscillator

$$|n\rangle = \frac{(a^{\dagger})^{n}}{\sqrt{n!}} |0\rangle \qquad \text{(normalized)}$$

$$I = \sum_{n=0}^{\infty} \frac{1}{n!} (a^+)^n |0\rangle\langle 0| a^n$$

- another, useful rep is in terms of coherent states $|z\rangle \propto e^{z a^+}|0\rangle$
- this is another complete set if we let z range over C
- actually overcomplete & not orthogonal but that doesn't matter

let's normalize $|z\rangle$:

$$\langle 0| e^{z^* a} e^{z a^+}|0\rangle = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(n!)^2} \langle 0| a^n (a^+)^n |0\rangle$$

$$= \sum_{0}^{\infty} \frac{|z|^{2n}}{n!} = e^{|z|^2}$$

so we write $|z\rangle = e^{-\frac{|z|^2}{2}} e^{z a^+}|0\rangle$, $\langle z|z\rangle = 1$

Consider $\int dz^* dz^{\phi} |z\rangle\langle z|$

[- normalize as $\int dz^* dz \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} d\,\mathrm{Re}\,z \int_{-\infty}^{\infty} d\,\mathrm{Im}\,z$]

$$= \int dZ^* dZ \; e^{-|Z|^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{Z^n}{n!} \frac{(Z^*)^m}{m!} (a^+)^n |0\rangle\langle 0| a^m$$

$$\int dZ^* dZ \; e^{-|Z|^2} Z^n Z^{*m} = 0 \text{ if } n \neq m \text{ by rotational}$$

invariance $\quad \propto \int_0^{2\pi} d\theta \; e^{i(n-m)\theta}$

for $n = m$

$$= \int_0^{\infty} du \; e^{-u} u^n \quad \left[\int \frac{dx\,dy}{\pi} = \int \frac{d\theta}{\pi} r\,dr \to 2\int r\,dr = \int du \right]$$

$$= n!$$

so $\int dZ^* dZ \, |Z\rangle\langle Z| = \sum_{n=0}^{\infty} \frac{(a^+)^n}{\sqrt{n!}} |0\rangle\langle 0| \frac{a^n}{\sqrt{n!}} = \sum |n\rangle\langle n| = I$

- consider generalization to a "fermionic oscillator"

- ie a Hilbert space of a single fermionic creation / annihilation operator $\hat{\psi}, \hat{\psi}^+$

[$\equiv C_{\vec{k}\sigma}$] - I put hats over operators temporarily to distinguish them from "Grassmann numbers" that I will introduce shortly

$I = |0\rangle\langle 0| + |1\rangle\langle 1| = |0\rangle\langle 0| + \hat{\psi}^{+}|0\rangle\langle 0|\hat{\psi}$

- "fermionic coherent states" is not labelled by a complex number Z as in bosonic case but instead by a Grassmann numbers $\psi, \bar{\psi}$
- these ~~obey~~ anti-commute

$\{\psi, \bar{\psi}\} = \{\psi, \psi\} = \{\bar{\psi}, \bar{\psi}\} = 0$ ie $\psi^2 = \bar{\psi}^2 = 0$

- this terminates $\sum |n\rangle\langle n|$ at $n=1$

N.B. different than creation / annihilation ops which obey $\{\hat{\psi}, \hat{\psi}^{+}\} = 1$

- Grassmann numbers defined to anti-commute with fermionic ops $\{\psi, \hat{\psi}\} = \{\psi, \hat{\psi}^{+}\} = 0$ etc & commute with bosonic operators
- also commute with ground state $\psi|0\rangle = |0\rangle\psi$
- we also define integrals over Grassmann numbers $\int d\psi = 0$, $\int d\psi \cdot \psi = -\int \psi \, d\psi = 1$
- this is complete "integral table" since $\psi^n = 0, n \geq 2$

and any function of ψ is defined by Taylor expansion $f(\psi) \equiv f(0) + f'(0) \cdot \psi$

- fermionic coherent states:

$$|\psi\rangle = |0\rangle - \psi|1\rangle = |0\rangle - \psi\hat{\psi}^{+}|0\rangle = (1 + \hat{\psi}^{+}\psi)|0\rangle$$

N.B. $|\psi\rangle$ is an "eigenstate" of $\hat{\psi}$ with "eigenvalue" ψ

Proof $\hat{\psi}|\psi\rangle = \hat{\psi}\hat{\psi}^{+}\psi|0\rangle = \psi|0\rangle$
$= \psi[|0\rangle - \psi|1\rangle]$
(since $\psi^2 = 0$)
$= \psi|\psi\rangle$

$\langle\bar{\psi}| \equiv \langle 0| - \langle 1|\bar{\psi}$ is a "bra" type coherent state, NB $\bar{\psi}$ is not the Hermitean conjugate of ψ - we don't try to define Hermitean conjugates of Grassmann numbers.

$\langle\bar{\psi}|$ is an "eigenstate" of $\hat{\psi}^{+}$ with "eigenvalue" $\bar{\psi}$

$\langle\bar{\psi}|\hat{\psi}^{+} = \langle\bar{\psi}|\bar{\psi}$ - Check!

(74)

Check "normalization" of fermionic coherent states $\langle \bar\psi | \psi \rangle = [\langle 0| - \langle 1|\bar\psi][|0\rangle - \psi|1\rangle]$

$= 1 + \bar\psi\psi = e^{\bar\psi\psi}$ [check sign]

[Can write it this way, for convenience since $e^{\bar\psi\psi} = \sum \frac{(\bar\psi\psi)^n}{n!}$, $(\bar\psi\psi)^n = 0$ for $n \geq 2$]

- so $|\psi\rangle$ is ~~not~~ unit normalized

- in fact its norm is not a c-number

NB $\int \bar\psi\psi \, d\psi \, d\bar\psi = 1$

$\int e^{-a\bar\psi\psi} d\bar\psi d\psi = -a\int \bar\psi\psi \, d\bar\psi d\psi = a.$

This is opposite of c-number ie

$\int e^{-a z^* z} dz^* dz = \frac{1}{a}$ [for $a > 0$]

3/11 $-a|z\rangle = z|z\rangle$,

$I = \int |\psi\rangle\langle\bar\psi| \, e^{-\bar\psi\psi} d\bar\psi d\psi$ - resolution of identity we seek

~~[be careful about order!]~~

Proof: $\int |\psi\rangle\langle\bar\psi| e^{-\bar\psi\psi} d\bar\psi d\psi$

$= \int [\,|0\rangle - \psi|1\rangle] [\langle 0| - \langle 1|\bar\psi\,] [1 - \bar\psi\psi] \, d\bar\psi d\psi$

$= -|0\rangle\langle 0| \int \bar\psi\psi \, d\bar\psi d\psi + |1\rangle\langle 1| \int \psi\bar\psi \, d\bar\psi d\psi = |0\rangle\langle 0| + |1\rangle\langle 1|$ ✓

- so formal integral over Grassmann numbers is a perfectly valid resolution of the identity op

- a related identity for any bosonic operator Ω is $\mathrm{tr}\,\Omega = \int \langle -\bar\psi|\Omega|\psi\rangle\, e^{-\bar\psi\psi}\, d\bar\psi\, d\psi$

  - note minus sign

Proof: $\int \langle -\bar\psi|\Omega|\psi\rangle\, e^{-\bar\psi\psi}\, d\bar\psi\, d\psi$

$$\cancel{= \int [\langle 0| + \bar\psi\langle 1|]\,[\,|0\rangle -}$$

$$\int [\langle 0| + \langle 1|\bar\psi]\,\Omega\,[\,|0\rangle - \psi|1\rangle]\, e^{-\bar\psi\psi}\, d\bar\psi\, d\psi$$

$$= \langle 0|\Omega|0\rangle \int e^{-\bar\psi\psi}\, d\bar\psi\, d\psi - \langle 1|\Omega|1\rangle \int \bar\psi\psi\, e^{-\bar\psi\psi}\, d\bar\psi\, d\psi$$

$$= \langle 0|\Omega|0\rangle \int -\bar\psi\psi\, d\bar\psi\, d\psi - \langle 1|\Omega|1\rangle \int \bar\psi\psi\, d\bar\psi\, d\psi$$

$$= \langle 0|\Omega|0\rangle + \langle 1|\Omega|1\rangle = \mathrm{tr}\,\Omega \checkmark$$

~~i.e. $\mathrm{tr}(\Omega\,|\psi\rangle\langle\bar\psi|)$~~

- we will, of course, want to generalise fermion coherent states to Hilbert space with many creation / annihilation ops $c_{\vec{k}\sigma}$