$$\frac{1}{\tau} = \frac{1}{(2\pi)^7} \frac{\varepsilon_1^2}{2\hbar v_F} k_F \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\frac{\theta}{2} |F(\theta,\phi)|^2$$

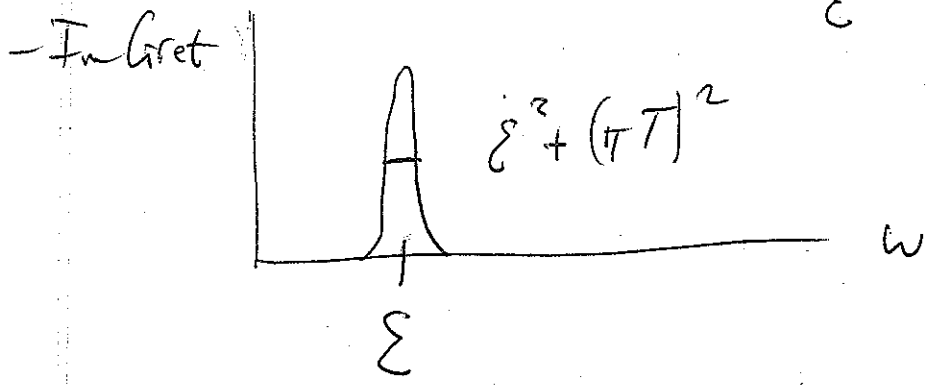
$$\propto \varepsilon_1^2$$

- at finite $T$, decay rate is larger because restriction $k_2 < 0$, $k_3, k_4 > 0$ are not strictly enforced but instead we get Fermi function

$$n_F(k_2)\,[1 - n_F(k_3)]\,[1 - n_F(k_4)]$$

$$\varepsilon_1^2 \to \varepsilon_1^2 + (\pi T)^2$$

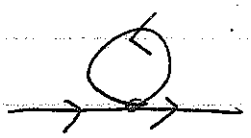
$$\frac{1}{\tau_1} \propto T^2 \quad \text{at} \quad \varepsilon_1 \to 0$$

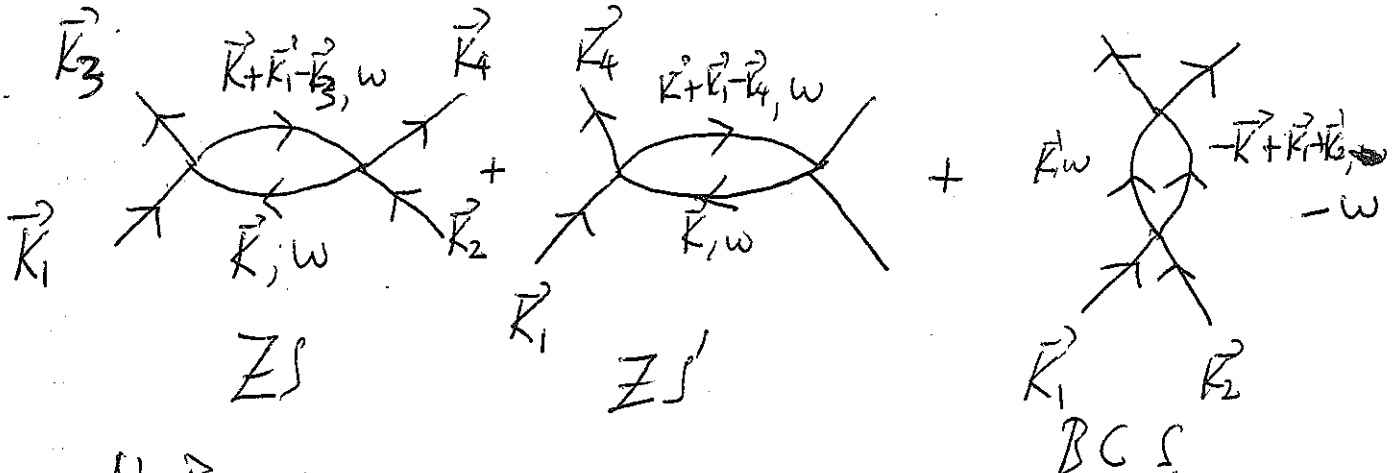



- looks more & more free-electron like

at $T, \mathcal{E} \to 0$

## 1 Loop RGin D=2+3

- "tadpole diagram"  contributes a $(\omega, k_0)$-independent term to self-energy
- Corresponds to a shift in chemical potential
$\delta \int = \delta\mu \int \bar{\psi}\psi$ - would shift density
- we add a "counter-term" to $\mu$ to keep density of interest
- not important at this stage but will come up again when we consider interaction energy of excited state
- Renormalization of interactions
- we get same 3 diagrams as in D=1




ZS ZS' BCS

N.B. we set $\omega = 0$ and $k = 0$ on all external lines, in order to calculate renormalization of marginal couplings only so all $|\vec{K}_i| = K_F$ -

- 2 types of choices for directions of $\vec{K}_i$
- F & V interactions

- renormalization of F - chose $\vec{K}_1 = \vec{K}_3$, $\vec{K}_2 = \vec{K}_4$ (up to a rotation of $\vec{K}_3, \vec{K}_4$ around $\vec{K}_1 + \vec{K}_2$ by $\phi_{12,34}$ in $D=3$)

- recall that internal momenta [$\vec{K}$ and $\vec{K}+\vec{K}_1-\vec{K}_3$ in ZS diagram] must be

in > momentum regia $\Lambda - d\Lambda < |k| < \Lambda$
where $K = K_F + k$.

- in particular, $k \neq 0$
- in most cases, these diagrams are
$\propto d\Lambda^2$ - so $\beta$-function vanishes [defined by $\lim_{d\Lambda \to 0} \frac{dF}{d\Lambda}$ ]
- Consider for example, ZS: $\vec{K}_1 - \vec{K}_4 = \vec{K}_1 - \vec{K}_2$
in generally a large momentum ~ of $O(K_F)$
- 2eq diagram
- same story for BCS diagram since
$\vec{K}_1 + \vec{K}_2$ is generally large for F-term
- what about ZS?
- in $D=2$, $\vec{K}_1 = \vec{K}_3$ for F-term

so we don't get $(d\Lambda)^2$ suppression

- however it gives zero for another reason

- same as in $D=1$:

$$\int_{-\infty}^{\infty} \frac{d\omega}{(i\omega - v k)^2} = 0$$

- here it is important that $k \neq 0$

- even if we take $|\vec{k}_1 - \vec{k}_3|$ as small value $\ll \Lambda$, $|\vec{k} + \vec{k}_1 - \vec{k}_3| - k_F$ will have same sign as $|\vec{k}| - k_F$ so $\omega$-integral vanishes

- in $D=3$, generally $|\vec{k}_1 - \vec{k}_3|$ is again large, so we get $(d\Lambda)^2$ factor

- so $\Lambda \frac{d}{d\Lambda} F(z_{1,3}, \phi_{1,2,3,4}) = 0$ to $O(F^2)$,

## ~~Renorm~~ Renormalization of V

- same diagrams, but now $\vec{K}_1 + \vec{K}_2 = 0$

$|\vec{K}_1 - \vec{K}_3|$ and $|\vec{K}_1 - \vec{K}_4|$ are generally large in this case, giving $(d\Lambda)^2$ factor in ZS, ZS'

- but BCS diagram gives no such $d\Lambda$ factor since internal momenta are $\vec{K}$ and $-\vec{K}$

- furthermore, $\omega$ integral is non-zero

$$\int_{-\infty}^{\infty} \frac{d\omega}{(i\omega - vk)(-i\omega - vk)} = \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 + (vk)^2} = \frac{\pi}{v|k|}$$

where $|\vec{k}| = |-\vec{k}| = k$, $\varepsilon(\vec{k}) = \varepsilon(-\vec{k}) = vk$

- like in $D=1$ but there is no second diagram to cancel it!

- letting ~~$\omega \to \infty$~~ $\hat{\Omega}$ be direction of internal momentum $\vec{K}$, we get $V(\hat{\Omega}_1, \hat{\Omega}$

at lower interaction vertex and $V(\hat{R}_2 \cdot \hat{R}_3)$

at upper vertex

$$\Lambda \frac{d}{d\Lambda} V(\theta_1 - \theta_3) = + \frac{1}{4\pi^2} \int_0^{2\pi} \frac{d\theta}{2\pi} V(\theta_1 - \theta) V(\theta - \theta_3)$$

in $D=2$

$$\text{or } \alpha + \int d^2\hat{R}\, V(\hat{R}_1 \cdot \hat{R}) V(\hat{R} \cdot \hat{R}_3)$$

in $D=3$

- expanding in circular (or spherical) harmonics $V(\theta) = \sum_l e^{-il\theta} V_l$

$\Lambda \frac{dV_l}{d\Lambda} = \frac{V_l^2}{4\pi}$ - like RG for Kondo coupling - marginally relevant if $V_l < 0$

corresponding to attractive interaction

- $V_l$ becomes large, even if initially small, at an exponentially small energy scale $\Delta \sim \Lambda D e^{-\frac{4\pi}{V_l}}$

- otherwise it renormalizes to 0

- in general, some $V_\ell$ might be positive some negative

- angular momentum channel for which $V$ gets large gives symmetry of Cooper pair wave-function in BCS theory

- s-wave, p-wave, d-wave etc.

- actually, a more careful analysis, keeping effects of higher order in $\Lambda/k_F$, shows that $V_\ell$'s at very large $\ell$, which have very small bare values, always become attractive and eventually renormalize to $\infty$

$\wedge$ $\frac{dV_\ell}{dl} = \frac{V_\ell^2}{4\pi} + C\, V(\ell=\pi)^2/\ell^4$, C depends on $k_F/\Lambda$

- Kohn-Luttinger instability

- but, in cases where this happens for large $\ell$, corresponding $\Delta$ is extremely small

And there is a large range of energies $\ll E_F$ where we may ignore $V$ and use Fermi liquid theory

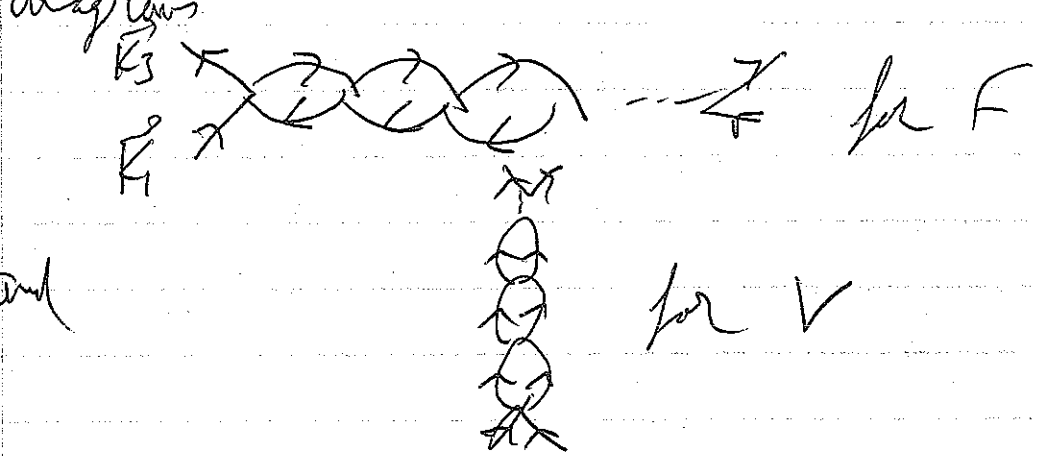
- some metals show no sign of superconductivity down to lowest measured temperature
- most metals that do exhibit superconductivity have $\Delta \sim T_c \leq 30\,K$ and Fermi liquid theory seems to apply for $T_c \ll E \ll E_F$
- also true of $He^3$ - $E_F = 5^\circ K$, $T_c = 1\,mK$
- high-$T_c$ cuprates appear to be an exception - $E_F$ small, $T_c$ large
- superconductivity in most metals is

believed to be due to electron-phonon interaction which I have not discussed

- this typically leads to ~~d~~ s-wave superconductivity - below energy scale $\omega_D$ we could use this electron-only approach with $g$, $V_0$ attractive

29/3 - Higher order β-functions

- most higher order diagrams have $(d\Lambda)^2$ factors for same reason as above [different momenta on internal lines] vanish
- only exceptions are ~~bubble~~ bubble diagrams



- bubbles for $F$ all vanish in RG calculation due to $\int_{-\infty}^{\infty} \frac{d\omega}{(i\omega - vk)^2} = 0$

- bubbles for $V$ are non-vanishing but they don't correspond to higher order terms in the $\beta$-function - just the series that give the leading logs:

$$V_\ell(\Lambda) = \frac{V_\ell^0}{1 + \frac{V_\ell^0}{4\pi} \ln \frac{\Lambda_0}{\Lambda}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{V_\ell^0}{4\pi} \ln \frac{\Lambda_0}{\Lambda}\right)^n$$

- so, we have exact $\beta$-function in narrow band limit, $\Lambda \ll K_F$

- Landau FLT corresponds to treating the effects of the marginal $F$ terms and ignoring the $V$ term

- expected to be a good approximation over energy range where $V_\ell(\Lambda)$'s are small

- in fact, "forward-scattering" part of

$F(Z_{1,2}, \phi_{12}; 3\phi) - F(Z_{1,2}, 0)$

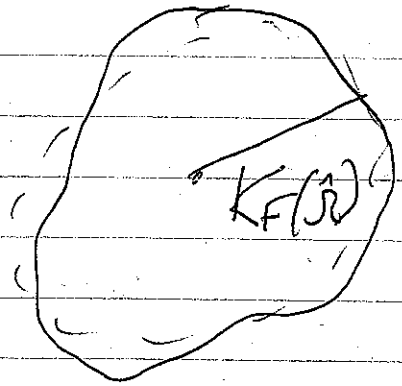
is much more important than $\phi_{12}; 3\phi = 0$

part.

- one way of seeing this is to calculate, as low energy
Landau did, energy to add a quasi-particle
to system when some are already present
- i.e. interaction energy between low
energy quasi-particles

- a convenient and elegant way of doing this
is to calculate the self-energy of a quasi-particle
coherent
in the presence of a hot-Fermi surface
which is deformed slightly due to the presence
of particles and holes

i.e. $k_F$ becomes a function of direction on the Fermi sphere




let $k_F(\hat{R}) = k_F + r(\hat{R})$ following Shankar.

- we can make this the ground state by making the chemical potential direction-dependent on the Fermi surface: - assume $|r(\hat{R})| < \Lambda$.

- quadratic term in $\hat{H} - \mu \hat{N}$ is modified to

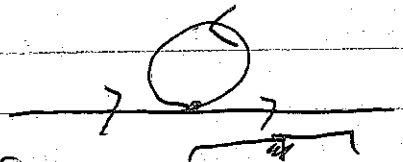
$$H_0' = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int \frac{d^2\hat{R}}{4\pi} \Psi^+ \left[ v^*(k - r(\hat{R})) \vec{\hat{n}} \cdot \vec{\nabla} \right] \Psi$$

- WB $v^*$ is a renormalized velocity - it picks up a renormalization during reduction of cut-off to a small value - doesn't renormalize in Landau Fermi liquid theory

$\mu$ is the chemical potential already present to keep Fermi surface at $k_F$ in the presence of interaction terms, $\propto F$

- using the F-interaction we can calculate the self-energy as a function of $r(\hat{n})$

- to $O(F)$ it comes from the tadpole diagram



$$\int \bar{\psi}(4)\psi(3)\bar{\psi}(2)\psi(1)\; F(\theta_{12}, \theta_{23})$$

$\langle \bar{\psi}(3)\psi(1)\rangle$ is only non-zero when $\vec{k}_3 = \vec{k}_1$

$\Rightarrow \theta_{12,34} = 0$ [recall that, in general $\vec{k}_3$ is rotated from $\vec{k}_1$ by angle $\theta_{12,34}$ around $\vec{k}_1 + \vec{k}_2$ axis]

$$\varepsilon'(k\hat{n}) = v^*(k - r(\hat{n})) - \int \frac{d\omega' dk' d\hat{n}'}{(2\pi)^4} F(\hat{n}\cdot\hat{n}', \omega')$$

$$\left[\frac{1}{i\omega' - v^*(k' - r(\hat{n}'))} - \frac{1}{i\omega' - v^* k'}\right]$$

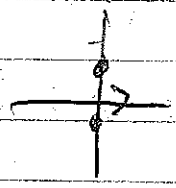
- $-2k'$ term is $O(F)$ shift in chemical potential, necessary so correction vanishes when $\Gamma = 0$

- note that $k'$ integral goes from $\int_{-\Lambda}^{\Lambda}$ not $\left(\int_{\Lambda-\Lambda}^{\Lambda} + \int_{-\Lambda}^{-\Lambda+\Lambda}\right)$

$$\propto \int_{-\infty}^{\infty} \frac{dw'}{2\pi} \left[ \frac{1}{iw' - v^*(k' - \Gamma(\hat{k}'))} - \frac{1}{iw - v^* k'} \right]$$

$$= + v^* \Gamma(\hat{k}') \int_{-\infty}^{\infty} \frac{dw'/2\pi}{[iw' - v^*(k' - \Gamma(\hat{k}'))][iw - v^* k']}$$

- this is only non-zero if $k'$ and $k' - \Gamma(\hat{k}')$ have opposite signs




- because $k'$ integral runs from $-\Lambda$ to $\Lambda$ and $|\Gamma(\hat{k}')| < \Lambda$ there will be a non-zero result after integrating over $k'$

w-integral $= -$ sgn $[\Gamma(\hat{k}')]$ when $k'$, $k' - \Gamma$ have opposite signs

- eg if $\Gamma > 0$, non-zero for $0 < k' < \Gamma$

- $k'$ integral gives $\Gamma(\hat{k}')$

$$\varepsilon(\hat{k}) = V^0 k + \int \frac{d\hat{k}'}{(2\pi)^3} F(\hat{k}-\hat{k}')\, r(\hat{k}')$$

$2^{nd}$ term represents int interacting quasi-particle at $k$ with other quasi-particles corresponding to deformation of Fermi surface $r(\hat{k}')$

- NB - only involves $F(\hat{k}-\hat{k}';\sigma)$

- I dropped $-V^0 r(\hat{k})$ term in order to measure $\varepsilon$ with respect to original Hamiltonian $H$, not $H'$

- we may relate $r(\hat{k})$ to number of particles added [or removed] per unit area on the Fermi surface

$$N = V \int \frac{d^3K}{(2\pi)^3} = \frac{V}{(2\pi)^3} \int d^2\hat{k} \int_0^{K_F(\hat{k})} dK\, K^2$$

$$N - N_0 \simeq \frac{V}{(2\pi)^3} K_F^2 \int d^2\hat{k}\, r(\hat{k}) = \int \frac{d^2\hat{k}}{4\pi} \delta n(\hat{k})$$

$$\delta n(\hat{k}) = \frac{V}{(2\pi)^3} K_F^2\, r(\hat{k})$$

- using this we can express energy $\delta$ in terms of this $\delta n$

- Going back to discrete sums over $\vec{K}$ at finite volume we can then write

$$E(\delta n) = \sum_{\vec{K}} V^* k\, \delta n(\vec{K}) + \frac{1}{2V} \sum_{\vec{K},\vec{K}'} \delta n(\vec{K})\, \bar{F}(\hat{K}\cdot\hat{K}')\, \delta n(\vec{K}') \quad (*)$$

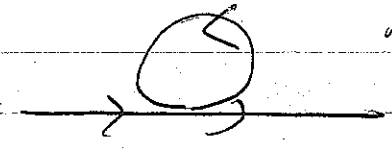
where $\bar{F}(\hat{K}\cdot\hat{K}') \propto F(\hat{K}\cdot\hat{K}', 0)$, $k = K - K_F$

$\delta n(\vec{K}) = 0$ or $1$ for $K > K_F$
$= 0$ or $-1$ for $K < K_F$
} assume only non-zero for $|K-K_F| < \Lambda$

- This is how Landau introduced F function, only depending on $\hat{K}\cdot\hat{K}'$, not $\phi$ which is 0
- F for $\phi \neq 0$ doesn't appear in $E(\delta n)$
- equivalently,

$$\frac{E(r(\hat{x}))}{V} = \frac{k_F^2}{(2\pi)^3}\left[ V^* \int r^2(\hat{x})\frac{d^2\hat{x}}{4\pi} + \frac{1}{2}\iint dx\,dx'\, r(x)\, F(x-x')\, r(x') \right]$$

- like deformation energy for a membrane
- quadratic in $r(\hat{x}) = k_F(\hat{x}) - k_F$

- it appears we have only evaluated self-energy to linear order in $F$ from 

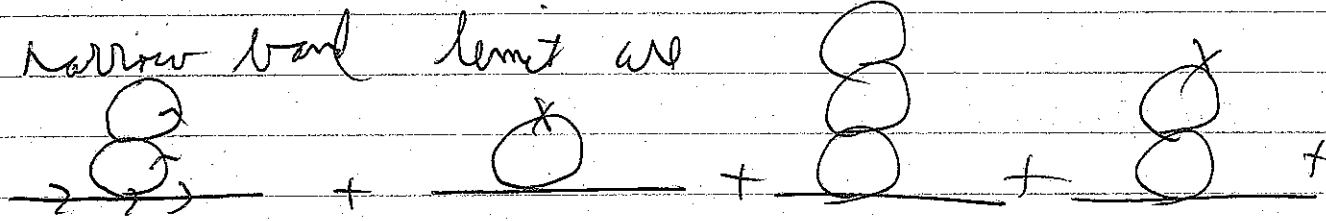
- but actually this is exact result in narrow band limit

- the technical reason is that to maintain the desired Fermi surface deformation $r(\hat{k})$ we must add an $O(F)$ "counter-term" to $H'$

$$\int \frac{dk\, d\Omega}{(2\pi)^3} \psi^+(k\Omega)\psi(k\Omega) \cdot \int \frac{d\Omega'}{(2\pi)^3} F(\hat{k}\cdot\hat{k}')\, r(k')$$

- represent by 

- only higher order terms in self-energy in narrow band limit are



and these all cancell each other

- Landau (1956) began his theory of Fermi liquids by observing that $\frac{1}{T} \propto E^2$ (at $T=0$) so low energy quasi-particles are free-electron-like [with some renormalized Fermi velocity $v^*$] and we can parameterize low energy states in terms of their number $f_n(\vec{k})$
- he then just proposed that the energy of a state with $f_n(\vec{k})$ particles added with momentum $\vec{k}$ should have the form of (*)
- i.e. there should be a pairwise interaction
- he realized that $\bar{F}(\hat{k}\cdot\hat{k}')$ was the forward scattering amplitude, at least for weak interactions
- in a later paper (1959) he realized that $\bar{F}$ should actually be given by a certain $\infty$ sum of Feynman diagrams - "the vertex part" in AGD - this was more or less equivalent to RG viewpoint

- a variety of low energy properties of Fermi liquids can be calculated in terms of $V^a$ and $F(\hat{k}\cdot\hat{k}')$: Zero sound, first sound, compressibility, specific heat, magnetic properties ...

- we will see some of these in student lectures

- a simple and famous result related $V^a$ to $F$ - as $V^a$ is not really an independent parameter - relationship is a result of Galilean invariance - i.e. that by boosting the velocities of all particles by $\delta\vec{v} = \delta\vec{p}/m$ [change of reference frame] just changes the energy by exactly $\frac{\vec{P}\cdot\delta\vec{p}}{m}$ where $m$ is the bare fermion mass

- this is exact for a system like $He^3$ with no lattice - no preferred reference frame

- this symmetry determines $\frac{V^a}{V}$ in terms of $F$

- we conventionally write $V^* = \frac{K_F}{m^*}$ [and $V = \frac{K_F}{m}$]

so, equivalently $\frac{m^*}{m}$ is determined by F

- to derive relationship consider an excited state with 1 quasi-particle sitting infinitesimally above the Fermi surface in $\hat{z}$ direction

- now boost in z direction by $\delta p$.

- then total momentum of the state before boost is $P = K_F \hat{z}$

2. $\delta E = \frac{\delta p K_F}{m}$ (Galilean invariance)

- alternatively we can calculate $\delta E$ using Fermi liquid Hamiltonian in terms of $V^*$ and F:

- kinetic energy term gives a contribution $\delta p \cdot V^* = \frac{\delta p K_F}{m^*}$

- there is another contribution from interaction term F - the boost distorts the Fermi surface.

- new Fermi surface is described by

$$|\vec{k} + \delta p \hat{z}| < K_F$$

$$K^2 + 2\delta p K z < K_F^2$$

$$K < K_F + \delta p \cos\epsilon$$

$$\Rightarrow \quad r'(\hat{r}) = \delta p \cos\epsilon = \delta p \, z$$

- from previous calculation Change in interaction

Energy is $\int \frac{d^3\Omega}{(2\pi)^3} F(\hat{r}\cdot\hat{z})\, \delta p \, z$

$$= \delta p \int \frac{d z}{4\pi} F(z) \, z$$

- $F(\hat{r}\cdot\hat{r}')$ is written as $\frac{2\pi^2 K_F}{m^*} \bar{\Phi}(\hat{r}\cdot\hat{r}')$

$$= 2\pi^2 \nu^* \bar{\Phi}(\hat{r}\cdot\hat{r}')$$

where $\bar{\Phi}$ is dimensionless and $\nu^*$ is a common factor in both terms in H

- expanding in spherical harmonics (Legendre polynomials)

$$\bar{\Phi}(z) = \sum_l \bar{\Phi}_l P_l(z)$$

$$\int_{-1}^{1} dz \, \bar{\Phi}_l(z) \cdot z = \frac{2}{3}\delta_{l,1} \qquad [P_1(z) = z]$$

Given $\frac{1}{m} = \frac{1}{m^*} + \frac{\bar{\Phi}_1}{3m^*}$

$$\Rightarrow \frac{m^*}{m} = 1 + \frac{1}{3}\bar{\Phi}_1$$

- $\Phi_1$ is completely determined by $\frac{\rho^*}{m} = \frac{V}{V^*}$
  & vice versa - reduces by 1, number of
  free parameters in Fermi liquid theory

- Answers to questions -

$\frac{1}{T} \propto \xi^2 \ln\left(\frac{c E_F}{\xi}\right)$ in $D=2$ where

$c$ is a constant of $O(1)$ - goes to zero
only slightly slower than in $D=3$

- general form of RG eqns in any model

$$\Lambda \frac{d\lambda_i}{d\Lambda} = f(\lambda_1, \lambda_2 \ldots, \Lambda)$$

- i.e. there could be dependence on running cut-off $\Lambda$
as well as on values of coupling constants at the
cut-off

- in many simple and important situations
there is no explicit dependence on $\Lambda$

- often there is some unlisted dependency which goes away as N is required — true of all examples discussed in course

- time for final meeting April 10–13.
  - see Doodle Poll
  - PS6 is posted
  - please fill out course evaluations on-line