(20A)

- now add interactions to fermion model

$$H_{INT} = u \int dx\, J_L(x)\, J_R(x)$$

[the only (exactly) marginal term]

bosonize:

$$H = \frac{1}{4}(\partial_+\phi)^2 + \frac{1}{4}(\partial_-\phi)^2 - \frac{u}{4\pi}\partial_+\phi\,\partial_-\phi$$

- still quadratic in ϕ!

- not worrying about effects of higher order in u: Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right] + \frac{u}{4\pi}\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right]$$

$$= \frac{1}{2}\left(1 + \frac{u}{2\pi}\right)\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right]$$

- we may restore standard form of $\mathcal{L}$ by rescaling φ:

let $\tilde{\varphi} = \sqrt{1 + \frac{u}{2\pi}}\;\varphi$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_t\tilde{\varphi})^2 - (\partial_x\tilde{\varphi})^2\right]$$

- ~~to pro~~ let $\tilde{\varphi} = e^{-\gamma}\varphi$

γ is a smooth function of u

$\gamma = -\frac{u}{4\pi} + O(u^2)$

- this method doesn't determine $\gamma(u)$ exactly - depends on irrelevant operator effects - but $\gamma(u)$ is known exactly for nearest neighbour tight binding model from Bethe ansatz solution
- to preserve commutation relations

$\tilde{\Pi} = e^{\gamma}\,\Pi$

- rescaling affects bosonization identities thus giving non-trivial Green's functions in fermion theory

- to get $\varphi_{R/L}$ in terms of $\tilde{\varphi}$, we first write $\varphi_{R/L}$ in terms of (φ, Π)

$$\partial_x \varphi_R = -\frac{1}{2}(\partial_t - \partial_x)\varphi$$

$$= -\frac{1}{2}(\Pi - \partial_x \varphi)$$

$$\varphi_R(x) = -\frac{1}{2}\int_{-\infty}^{x} \Pi(x')\,dx' + \frac{1}{2}\varphi(x)$$

similarly $\varphi_L(x) = +\frac{1}{2}\int_{x} \Pi + \frac{1}{2}\varphi$

$$\varphi_{R/L} = \mp \frac{1}{2} e^{-\gamma} \int \tilde{\Pi} + \frac{1}{2} e^{\gamma} \tilde{\varphi}$$

- re-write in terms of $\tilde{\varphi}_R$, $\tilde{\varphi}_L$

$\tilde{\varphi} = \tilde{\varphi}_R + \tilde{\varphi}_L$,

$$\varphi_{R/L} = \mp \frac{1}{2} e^{-\gamma}\left(-\tilde{\varphi}_R + \tilde{\varphi}_L\right) + \frac{1}{2} e^{\gamma}\left(\tilde{\varphi}_R + \tilde{\varphi}_L\right)$$

$$\begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix} = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cancel{\sinh\gamma}\ \cosh\gamma \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_R \\ \tilde{\varphi}_L \end{pmatrix}$$

- this preserves commutation relations

$$[\varphi_R(x), \varphi_R(y)] = -[\varphi_L(x), \varphi_L(y)] = \frac{i}{4}\,\mathrm{sgn}(x-y)$$

$$\psi_R \propto e^{-i\sqrt{4\pi}\left[\cosh\gamma\ \tilde{\varphi}_R + \sinh\gamma\ \tilde{\varphi}_L\right]}$$

$\tilde{\varphi}_R$, $\tilde{\varphi}_L$ are just free boson fields with normal Green's functions

$$\langle \psi_R^+(t,x)\,\psi_R(0,0)\rangle \propto$$

$$\langle e^{i\sqrt{4\pi}\cosh\gamma\,\tilde{\varphi}_R(t,x)}\ e^{-i\sqrt{4\pi}\cosh\gamma\,\tilde{\varphi}_R(0,0)}\rangle$$

$$\cdot \langle e^{i\sqrt{4\pi}\sinh\gamma\,\tilde{\varphi}_L(t,x)}\ e^{-i\sqrt{4\pi}\sinh\gamma\,\tilde{\varphi}_L(0,0)}\rangle$$

$$\propto \left[\frac{1}{t-x-i\eta}\right]^{\cosh^2\gamma}\left(\frac{1}{t+x-i\eta}\right)^{\sinh^2\gamma}$$

- Fourier transform to get spectral density

check at $\gamma=1$ (free fermion)

$$G(\omega,k)\propto \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx\ \frac{e^{i(\omega t+kx)}}{(t-x-i\eta)}$$

let $t_\pm = t \pm x$

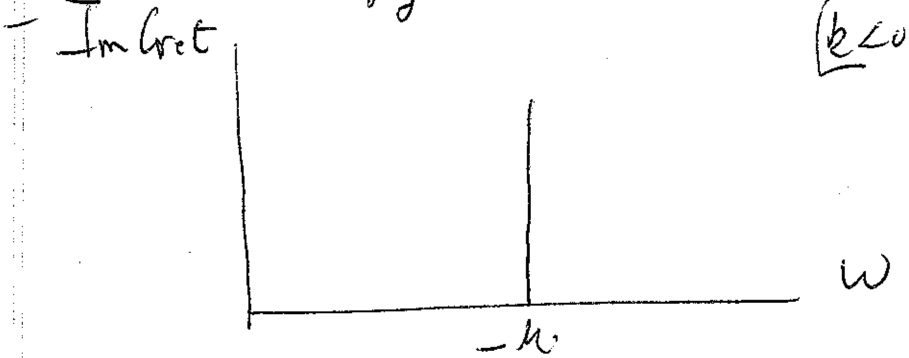
$$G(\omega,k) \propto \int_{-\infty}^{\infty} dt_- \int_{-\infty}^{\infty} dt_+\ \frac{e^{i\left[t_-\left(\frac{\omega-k}{2}\right)+t_+\left(\frac{\omega+k}{2}\right)\right]}}{t_- - i\eta}$$

$\propto \delta(\omega+k)\,\theta(\omega-\epsilon)$

$= \delta(\omega+k)\,\theta(\omega)$ [since $k<0$]

$\psi_R(k)$ creates a hole with momentum $k<0$

and energy $|k| = -k$.

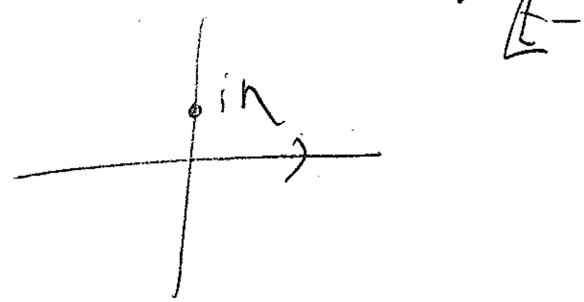


Fermi liquid theory result - spectral function

has a pole at $\omega = \mathcal{E}(k)$ as $T \to 0$, $\mathcal{E}(k) \to 0$

NB - t_- integral has simple pole

at $t_- = i\eta$



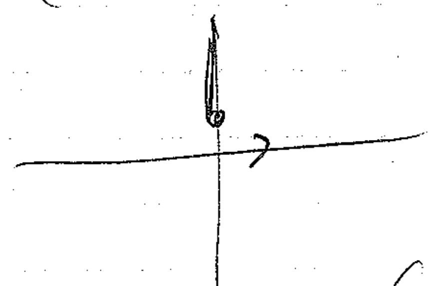
- close contour in upper half plane

for $\omega_- > 0$

at $\gamma \neq 0$ $t_\pm$ integrals have branch cuts

$$\int dt_- \frac{e^{i t_- \frac{(\omega - k)}{2}}}{(t_- - i\eta)^{\cosh^2\gamma}}$$

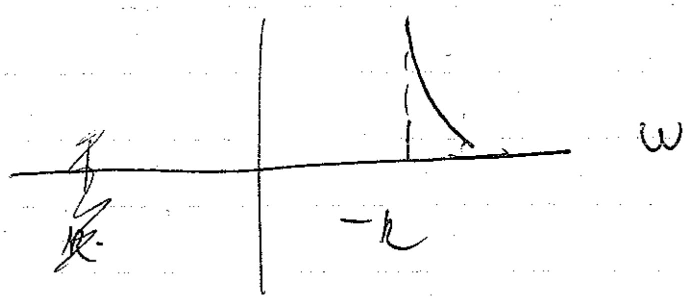
$\lfloor t_-$



$$\propto \theta(\omega - k)\,(\omega - k)^{(\cosh^2\gamma - 1)}$$

$$G(\omega, k) \propto \frac{\theta(\omega + k)\,\theta(\omega - k)\,(\omega - k)^{\cosh^2\gamma - 1}}{(\omega + k)^{1 - \sinh^2\gamma}}$$

$\lfloor k < 0$



- pole is replaced by power law singularity

$$\propto \frac{1}{(\omega + k)^{\eta}}, \quad \eta = 1 - \sinh^2\gamma = 1 - \left(\frac{u}{4\pi}\right)^2$$

- a non-fermi liquid - no pole at $T = \varepsilon = 0$

- this has experimental consequences for density of states measured in ~~Ten~~ tunnelling experiments among other things

<u>Interacting Fermions in $D=2$ and 3</u>

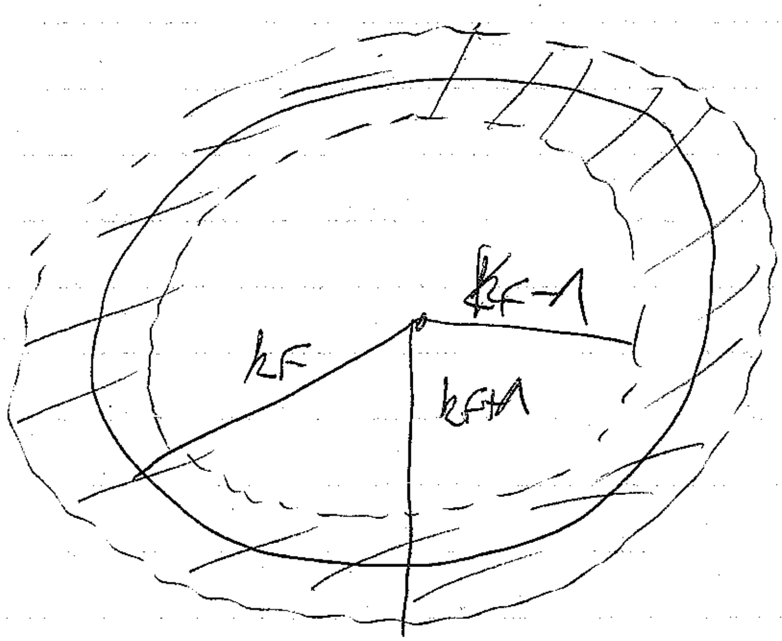
- final part of course - much of material presented by students
- I will follow Shankar closely
- complications arise in RG approach due to Fermi surface
- begin with simplest case: $D=2$ and circular Fermi surface

$$\varepsilon(\vec{K}) = \frac{K^2}{2m} - \mu$$

- after initial stages of RG we have only a narrow band of states remaining around Fermi "surface" ie Fermi circle of radius k_F

- only remaining states have

$$k = (K - K_F) \quad \text{obeying} \quad -\Lambda < K - K_F < \Lambda$$



$$E = v_F k \qquad (\text{drop } F \text{ subscript})$$

- let θ measure direction of $\vec{K} = (K_F + k)(\cos\theta, \sin\theta)$
- $\psi(\omega \vec{K}) \to \psi(\omega k \theta)$

$$S_0 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \bar{\psi}(\omega k \theta)(i\omega - vk)\psi(\omega k \theta)$$

[we absorbed a factor of $\sqrt{K_F}$ into $\psi, \bar{\psi}$]

- this is invariant under same RG transf as in $D=1$:

$\Lambda \to \frac{\Lambda}{s}$ so $k = \frac{\tilde{k}}{s}$ $\quad |\tilde{k}| < \Lambda$

$\omega = \frac{\tilde{\omega}}{s}$ $\quad \psi = s^{3/2}\psi'$

- this ressembles an infinite number of 1-dimensional relativistic fermion fields labelled by angle θ on Fermi surface

N.B.— this looks nothing like a relativistic

Fermion model in $D=2$

[we only get that in special cases like graphene with unusual dispersion relation]

- instead this is like a complicated 1D relativistic model

- general quartic interaction term in low energy Hamiltonian [only keeping states near Fermi circle]

can be written:

$$\delta S_4 = \frac{1}{(2!)^2} \int_{K\omega\sigma} \bar{\Psi}(4)\bar{\Psi}(3)\Psi(2)\Psi(1) u(4,3,2,1)$$

where $\Psi(1) = \Psi(\omega_1 k_1 \sigma_1)$ etc.

and, after eliminating $\int \frac{d\omega_4 dk_4}{(2\pi)^c}$ integral with energy and momentum conserving

δ - function:

$$\int_{K\omega\theta} \equiv \prod_{i=1}^{3} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \int_0^{2\pi} \frac{d\theta_i}{2\pi}\, \Theta(\Lambda - |k_4|)$$

here $k_4 \equiv |\vec{K}_1 + \vec{K}_2 - \vec{K}_3| - K_F$

i.e. $\int dk_4$ is ~~only~~ restricted to $|k_4| < \Lambda$

so $\delta(\vec{K}_1 + \vec{K}_2 - \vec{K}_3 - \vec{K}_4)$ is only non-zero

when $|\vec{K}_1 + \vec{K}_2 - \vec{K}_3|$ is in this narrow range.

$\Theta(\Lambda - k_4)$ transforms in a complicated way

22/3 under $\Lambda \to \frac{\Lambda}{s}$, $k = \frac{\tilde{k}}{s}$ because

$\vec{K}_1 + \vec{K}_2 - \vec{K}_3 = (K_F + k_1)\hat{\Omega}_1 + (K_F + k_2)\hat{\Omega}_2$

$-(K_F + k_3)\hat{\Omega}_3$ where $\hat{\Omega}_i = (\cos\theta_i, \sin\theta_i)$

$k_i = \tilde{k}_i / s$

$\vec{K}_i \to \left(K_F + \frac{\tilde{k}_i}{s}\right)\hat{\Omega}_i$ - we do not get a simple rescaling of $\vec{K_i}$ or, therefore of

k_4

- fortunately, $\Theta(\Lambda - |k_4|)$ also implies that

$\int d\theta_1 d\theta_2 d\theta_3$ integrals can be greatly simplified - for most choices of these 3 angles $\Theta(\Lambda - |k_4|) = 0$

- there are 2 possibilities when $\Lambda \ll K_F$

(1) $\vec{K_1} = \vec{K_3}$, $\vec{K_2} = \vec{K_4}$,
or $\vec{K_1} = \vec{K_4}$, $\vec{K_2} = \vec{K_3}$

(these 2 correspond to same term in S_{eff} due to Fermi statistics)

- $\vec{K_1}$ and $\vec{K_2}$ are arbitrary vectors of length $= K_F$

(2) $\vec{K_1} = -\vec{K_2}$, $\vec{K_3} = -\vec{K_4}$ [typo in (350), (351)]
[$\vec{K_1}$, $\vec{K_3}$ arbitrary]

$$\delta S_F = \int \prod_{i=1}^{4} \frac{dk_i d\omega_i}{(2\pi)^2} (2\pi)^2 \delta(\omega_1+\omega_2-\omega_3-\omega_4)\,\delta(k_1+k_2-k_3-k_4)$$

$$\int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi}\, \bar\Psi(k_4,\omega_4,\theta_1)\, \bar\Psi(k_3,\omega_3,\theta_1+\theta)$$

$$\Psi(k_2,\omega_2,\theta_1)\, \Psi(k_1,\omega_1,\theta_1+\theta)\, F(\theta)$$

N.B. F only depends on θ - ~~phase~~ angle between $\vec{K}_1$ and $\vec{K}_2$, not angle of $\vec{K}_1$ itself - due to rotational invariance

- forward scattering process - initial & final $\vec{K}$s the same

$F(\theta)$ is an arbitrary coupling function

- ~~infinite number of coupling constants~~

$$F(\theta) = \sum_m F_m e^{im\theta}$$

- could be additional terms with this angle dependence but with powers of K_i, ω_i
- all irrelevant as before

$$\delta S_\psi = \int \prod_{i=1}^{4} \frac{dk_i d\omega_i}{(2\pi)^2} (2\pi)^2 \delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(k_1+k_2-k_3-k_4)$$

$$\bar\Psi(k_4,\omega_4,-\theta_3)\,\bar\Psi(k_3,\omega_3,\theta_3)\,\Psi(k_2,\omega_2,-\theta_1)\,\Psi(k_1,\omega_1,\theta_1)$$

$$V(\theta_1-\theta_3)$$

- again only angle $\theta_1-\theta_3$ enters V not $\theta_1+\theta_3$ — rotating θ_1, θ_3 by same angle has no effect on V by rotational invariance

- a 2nd infinite set of coupling constants

- as an example we can calculate $F(\theta)$, $V(\theta)$ explicitly for nearest neighbor tight-binding model at low density where Fermi surface is approximately circular

$$\epsilon(\vec{k}) = -2t[\cos k_x a + \cos k_y a] - \mu$$

$$\simeq -4t - \mu + t a^2 \vec{k}^2$$

$$t a^2 k_F^2 \simeq 4t + \mu \quad (\text{for } \mu \simeq -4t)$$

$$\delta H \propto \sum_{\vec{R}, \vec{e}} \psi^\dagger_{\vec{R}} \psi_{\vec{R}} \psi^\dagger_{\vec{R}+\vec{e}} \psi_{\vec{R}+\vec{e}}$$

$\vec{R} \in$ lattice, $\vec{e} = \pm a(1,0), \pm a(0,1)$

$$\delta H \propto \sum_{\vec{R} e} \psi^\dagger_{\vec{R}} \psi^\dagger_{\vec{R}+\vec{e}} \psi_{\vec{R}} \psi_{\vec{R}+\vec{e}}$$

$$= \frac{1}{4} \sum_{\vec{R} \vec{e}} [\psi^\dagger_{\vec{R}} \psi^\dagger_{\vec{R}+\vec{e}} - \psi^\dagger_{\vec{R}+\vec{e}} \psi^\dagger_{\vec{R}}]$$
$$[\psi_{\vec{R}} \psi_{\vec{R}+\vec{e}} - \psi_{\vec{R}+\vec{e}} \psi_{\vec{R}}]$$

$$\propto \int \prod_{i=1}^{4} d^2 k_i \quad \delta^2(k_1 + k_2 - k_3 - k_4) \, \bar{\psi}^\dagger_4 \psi^\dagger_3 \psi_2 \psi_1$$

$$\sum_{\vec{e}} [e^{-i\vec{k}_3 \cdot \vec{e}} - e^{-i\vec{k}_4 \cdot \vec{e}}][e^{i\vec{k}_2 \cdot \vec{e}} - e^{i\vec{k}_1 \cdot \vec{e}}]$$

N.B. anti-symmetric under

$\vec{k}_3 \leftrightarrow \vec{k}_4$ $\vec{k}_1 \leftrightarrow \vec{k}_2$

$$= 2\left[\cos(K_2-K_3)_x a + \cos(K_1-K_4)_x a\right.$$
$$\left. -\cos(K_3-K_1)_x a - \cos(K_4-K_2)_x a\right] + x \leftrightarrow y$$

- expand for small K_F:

$$\propto (\vec{K}_2-\vec{K}_3)^2 + (\vec{K}_1-\vec{K}_4)^2 - (\vec{K}_3-\vec{K}_1)^2 - (\vec{K}_2-\vec{K}_4)^2.$$
$$\propto \vec{K}_2\cdot\vec{K}_3 + \vec{K}_1\cdot\vec{K}_4 - \vec{K}_3\cdot\vec{K}_1 - \vec{K}_2\cdot\vec{K}_4$$
$$= (\vec{K}_2-\vec{K}_1)\cdot(\vec{K}_4-\vec{K}_3)$$

- Could have written down by inspection
  odd under $(\vec{K}_1 \leftrightarrow \vec{K}_2)$, $(\vec{K}_3 \leftrightarrow \vec{K}_4)$
  and quadratic

- for F term $\vec{K}_3 = \vec{K}_1$, $\vec{K}_4 = \vec{K}_2$

$$\Rightarrow F \propto (\vec{K}_1-\vec{K}_2)^2 \propto (\hat{k}_1-\hat{k}_2)^2 = 2(1-\cos\theta_{12})$$

V term: $\vec{K}_1 = -\vec{K}_2$, $\vec{K}_3 = -\vec{K}_4$

$$V \propto \vec{K}_1\cdot\vec{K}_3 \propto \cos\theta_{13}$$