The best way to understand low energy behavior of interacting fermions in $D=1$ is to use <u>Bosonization</u>

- show that interacting fermionic model [arbitrary U_0] is equivalent to a non-interacting bosonic model [Hamiltonian independent of U but U appears in way fermionic operators are represented as bosonic ops]
- this equivalence is exact in quantum field theory where we take $\Lambda \to \infty$
- it is approximate in condensed matter applications - valid at energy scales small compared to original cut-off
- still very useful and gives exact description of low energy RG fixed point action

- we begin by studying properties of a free massless, Lorentz invariant [with (-) V_F] boson model

- we will then be able to establish connection with (interacting) fermion model

Hamiltonian $H = \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\Pi(x)^2 + V_F^2 \left(\frac{d\varphi}{dx}\right)^2 \right]$

where $[\Pi(x), \varphi(y)] = -i\delta(x-y)$

$[\varphi, \varphi] = [\Pi, \Pi] = 0$

NB - this is a <u>commutator</u> not an anti-commutator

$\Pi(x)$ is conjugate momentum operator

- such a Hamiltonian arises in condensed matter physics for acoustic phonons in 1D

- then $\varphi(x) \sim$ displacement of atom at x,

$\Pi(x) \sim$ momentum of atom at x

- we can write corresponding Lagrangian in classical model by usual transformation

$$\partial_t \varphi(x) = \frac{\delta H}{\delta \Pi(x)} = \Pi(x)$$

$\mathcal{L} = \Pi \, \partial_t \varphi - H$ [where H is Hamiltonian density, $\mathcal{L}$ is Lagrangian density]

$$\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2} V_F^2 (\partial_x \varphi)^2$$

- this is Lorentz invariant under

$$\begin{pmatrix} V_F t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} V_F t \\ x \end{pmatrix}$$

and $\varphi(t,x) \to \varphi(t', x')$

- same Lorentz transformation as in fermion model but ψ_L, ψ_R transform non-trivially, φ is a Lorentz scalar.

- can check that $\partial_t^2 - V_F^2 \partial_x^2 = \partial_{t'}^2 - V_F^2 \partial_{x'}^2$ using $\cosh^2\alpha - \sinh^2\alpha = 1$

- we can expand $\varphi(x,t)$ in creation/annihilation operators ~~in~~ labelled by wave-vectors [Fourier transform] so as to give correct canonical commutators

- like Harmonic oscillator $X \sim a + a^\dagger$, $p \sim i(a - a^\dagger)$

- we work at ∞ length, so wave-vector is continuous and normalize commutators, analogous to fermionic ~~x~~ case, as

$$[\, a_k ,\; a_{k'}^\dagger \,] = 2\pi\,\delta(k - k')$$

then
$$\varphi(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2v_F|k|}} \Big[e^{-i[v_F|k|t - kx]}\, a_k + e^{i[v_F|k|t - kx]}\, a_k^\dagger \Big]$$

N.B. - we let k-integrals extend to $\pm\infty$ here as in quantum field theory but in C.M.P. applications there will be a cut off Λ of order the

(1 + 1)

- cut off in corresponding fermionic theory
- to check normalization, calculate commutator:

$$\Pi(x) = \partial_t \varphi = -i \int \frac{dk}{2\pi} \sqrt{\frac{v_F|k|}{2}} \left[e^{-i[v_F|k| - kx]} a_k - h.c. \right]$$

$$[\varphi(x), \Pi(y)] = -i \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-y)} = -i\delta(x-y)$$

check yourself please!

Can write H in k-space as

$$H = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, a_k^{+} a_k \, v_F|k| \quad \text{- check}$$

~~Schroedinger~~ → ~~p.~~ Heisenberg rep. time-dependent operators are $a_k(t) = e^{iHt} a_k e^{-iHt}$

$$= e^{-iv_F|k|t} a_k$$

since acting with a_k on any state lowers energy by $v_F|k|$

- imaginary time: $a_k(\tau) = e^{H\tau} a_k e^{-H\tau} = e^{-v_F|k|\tau} a_k$

- imaginary time; space field $\varphi(T,X)$ has expansion

$$\varphi(T,X) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2V_F|k|}} \left[e^{-V_F|k|T+ikX} a_k + e^{V_F|k|T-ikX} a_k^{+} \right]$$

→ p 183

- Consider Matsubara G-F. (at zero temperature)

$$T\langle 0|\varphi(T,x)\,\varphi(T',x')|0\rangle \equiv G(T-T', x-x')$$

$$= \langle 0|\varphi(T,x)\,\varphi(T',x')|0\rangle \quad (\text{for } T>T')$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi\, 2V_F|k|}\, e^{-V_F|k|(T-T')+ik(x-x')}$$

- integral diverges at $k \to 0$!!

- GF is not finite - infrared problem - finite Λ doesn't help

- fortunately, G.F.'s of fermionic operators only involve differences of GF

$$G(T,x) - G(0,0) = \int_{-\infty}^{\infty} \frac{dk}{4\pi V_F|k|} \left[e^{-V_F|k|T+ikx} - 1 \right]$$

- the integral converges at $k \to 0$ but diverges at $|k| \to \infty$ [integral of first term converges there but not integral of second term]
- this is OK because we actually have an ~~with~~ ultra-violet cut-off, Λ

[in field theory approach we can take $\Lambda \to \infty$ eventually]

- before evaluating $G(T,x) - G(0,0)$, let's return to mode expansion for $\varphi(t,x)$ and observe that is decomposes into left and right moving terms, similar to fermionic case: $\varphi(t,x) = \varphi_R(v_F t - x) + \varphi_L(v_F t + x)$

where $\varphi_R(t,x) = \int_0^\infty \frac{dk}{2\pi\sqrt{2 v_F k}} \left[e^{-i(v_F t - x)k} a_k + h.c. \right]$

$\varphi_L(t,x) = \int_{-\infty}^0 \frac{dk}{2\pi\sqrt{-2 v_F k}} \left[e^{i(v_F t + x)k} a_k + h.c. \right]$

[or cut off integrals at $|k| < \Lambda$]

Similarly, imaginary time field decomposes

$\varphi(\tau,x) = \varphi_R(v_F\tau - ix) + \varphi_L(v_F\tau + ix)$

- Green's function also decomposes.

$G(\tau,x) = G_R(v_F\tau - ix) + G_L(v_F\tau + ix)$

where $G_R(\tau,x) = T\langle \varphi_R(\tau,x)\, \varphi_R(0,0)\rangle$ etc.

$$G_R(v_F\tau - ix) - G_R(0) = \int_0^{\Lambda} \frac{dk}{4\pi v_F k}\left[e^{-(v_F\tau - ix)k} - 1\right]$$

at $x=0$

$$G_R(v_F\tau) - G_R(0) = \int_0^{\Lambda} \frac{dk}{4\pi v_F k}\left[e^{-v_F\tau k} - 1\right]$$

[for $\tau > 0$]

$$= \int_0^{\Lambda v_F \tau} \frac{du}{4\pi v_F u}\left[e^{-u} - 1\right]$$

- this is function of $\Lambda v_F \tau$ only -

"exponential - integral function"

- see Wikipedia or MathWorld

- we are only interested in case $\Lambda v_F \tau \gg 1$

- i.e. we study energies small compared to

$v_F\Lambda$, and correspondingly times long compared to $1/v_F\Lambda$

- in this limit we may approximate

$$G_R(v_F\tau) - G_R(0) = -\int_{\frac{1}{v_F\tau}}^{\Lambda} \frac{dk}{4\pi v_F k}$$

$$= -\frac{1}{4\pi v_F} \ln(\Lambda v_F \tau)$$

using $e^{-v_F\tau k} \approx 1$ for $v_F\tau k < 1$

≈ 0 for $v_F \tau k > 1$

- including $x \neq 0$ we get analytic continuation of exponential integral function into complex plane

- for $\Lambda |v_F\tau - ix| \gg 1$

$$G_R(v_F\tau - ix) - G_R(0) \approx -\frac{1}{4\pi v_F} \ln \Lambda(v_F\tau - ix)$$

- similarly left-moving term is

$$G_L(v_F\tau + ix) - G_L(0) \approx -\frac{1}{4\pi v_F} \ln \Lambda(v_F\tau + ix)$$

so entire GF is

$$G(\tau,x) - G(0,0) = -\frac{1}{4\pi V_F} \ln \Lambda^2 (V_F^2 \tau^2 + x^2)$$

- these expressions for Green's functions were derived for $\tau > 0$ but they can be seen to be correct at $\tau < 0$ also - check!

- no discontinuity at $\tau = 0$ for bosonic GF since $[\varphi(0,x), \varphi(0,y)] = 0$

- let us work in units where $\boxed{V_F = 1}$ to simplify notation

- we can re-insert it later

- we define "light cone derivatives"

$\partial_+ = \partial_t + \partial_x, \quad \partial_- = \partial_t - \partial_x$ [real time]

$$\partial_+ \varphi(t,x) = \partial_+ \varphi_R(t-x) + \partial_+ \varphi_L(t+x)$$

$$= \partial_+ \varphi_L = 2\,\partial_x \varphi_L$$

$$\partial_- \varphi = \partial_- \varphi_R = -2\,\partial_x \varphi_R$$

(187)

$$\langle \partial_- \varphi(t,x)\, \partial_- \varphi(0,0) \rangle$$
$$= \int_0^\infty \frac{dk}{4\pi k} (2k)^2 e^{-ik(t-x)}$$
$$= \int_0^\infty \frac{dk}{\pi} k\, e^{-ik(t-x)}$$

this has no infra-red divergence and is conditionally convergent at large k.

- if we insert a convergence factor $e^{-k\eta}$ in integrand $[\eta \to 0^+]$

we get $\langle \partial_- \varphi(t,x)\, \partial_- \varphi(0,0) \rangle$
$$= i\,\partial_t \int_0^\infty \frac{dk}{\pi} e^{-ik(t-x-i\eta)}$$
$$= \frac{i}{\pi} \partial_t \frac{1}{i(t-x-i\eta)}$$
$$= -\frac{1}{\pi (t-x-i\eta)^2}$$

- more correctly, instead of convergence factor $e^{-\eta k}$ we should ~~set~~ cut off integral at $k < \Lambda$ but this gives similar expression at large $|t-x|$ with $\eta \to \frac{1}{\Lambda}$

- NB - this is true for t positive or negative for non-time-ordered G.F.

- similarly $\langle \partial_+ \varphi(t,x)\, \partial_+ \varphi(0,0) \rangle = -\dfrac{1}{\pi(t+x-i\eta)^2}$

(just take $x \to -x$ - parity transformation)

- equal time commutators of $\partial_\pm \varphi$ are important for establishing bosonization

$$[\partial_+\varphi(0,x), \partial_+\varphi(0,y)]$$

$$= [\partial_t\varphi + \partial_x\varphi, \partial_t\varphi + \partial_y\varphi]$$

$$= [\partial_t\varphi(x), \partial_y\varphi(y)] + [\partial_x\varphi(x), \partial_t\varphi(y)]$$

$$= -i\frac{d}{dy}\delta(x-y) + i\frac{d}{dx}\delta(x-y) = 2i\frac{d}{dx}\delta(x-y)$$

Check: $-\dfrac{1}{\pi(x-y-i\eta)^2} + \dfrac{1}{\pi(y-x-i\eta)^2}$

$$= \frac{d}{dx}\left[\frac{1}{\pi(x-y-i\eta)} - \frac{1}{\pi(x-y+i\eta)}\right]$$

$$= \frac{d}{dx} \frac{2i\eta}{\pi[(x-y)^2+\eta^2]}$$

$$= 2i \frac{d}{dx} \delta(x-y)$$

similarly

$$[\partial_- \varphi(0,x), \partial_- \varphi(0,y)]$$

$$= -2i \frac{d}{dx} \delta(x-y)$$

[opposite sign]

PS5 - on-line

15/3 - now compare Green's functions, commutators and Hamiltonians of ~~a~~ free boson & free fermion models [we will extend to interacting fermion model later]

- I will show that we can identify $\psi_L^+\psi_L \sim \partial_+ \phi$

$\psi_R^+\psi_R \sim \partial_- \phi$

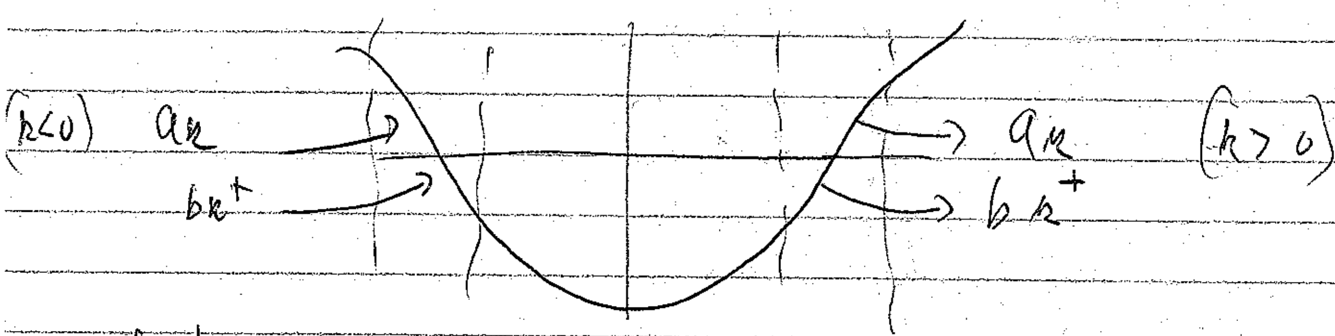
- mode expansion:

[other approaches to deriving Bosonization - see Mahan or Haldane, Phys Rev Lett 47, 1840 (1981)]

$$\psi_R(x) = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi}\, \psi_R(k)\, e^{ikx}$$

$$= \int_{0}^{\Lambda} \frac{dk}{2\pi} \left[a_k e^{ikx} + b_k^{+} e^{-ikx} \right]$$

where $\left.\begin{array}{l} a_k = \psi_R(k) \\ b_k^{+} = \psi_R(-k) \end{array}\right\} k > 0$



$b_k^{+} = \psi(-k)$ creates a hole at wave-vector

$k = -k \quad (k>0)$

- "positron" in particle physics

NB $\{a_k, a_{k'}^{+}\} = \{b_k, b_{k'}^{+}\} = 2\pi\,\delta(k-k')$

$\{\psi_R(x), \psi_R^{+}(y)\} = \delta_\Lambda(x-y)$

[ie a Dirac δ-function broadened by $\frac{1}{\Lambda}$]

$a_k(t) = e^{-ikt} a_k$

$$\psi_R(t,x) = \int_0^{\Lambda} \frac{dk}{2\pi} \left[a_k e^{-ik(t-x)} + b_k^{+} e^{ik(t-x)} \right]$$

similarly

$$\psi_L(t,x) = \int_{-\Lambda}^{0} \frac{dk}{2\pi} \left[a_k e^{ik(t+x)} + b_k^{+} e^{-ik(t+x)} \right]$$

- fermionic GF [time-correlation function]

$\langle 0 | \psi_R^{+}(t,x) \, \psi_R(0,0) | 0 \rangle$

$$= \int_0^{\Lambda} \frac{dk}{2\pi} \, e^{-ik(t-x)} \qquad [a_k \text{ on left}]$$

N.B. $\quad a_k |0\rangle = b_k |0\rangle = 0$

$$\langle 0 | a_k^{\prime +} a_k | 0 \rangle = 2\pi \delta(k-k')$$

- convenient again to replace cut off Λ by convergence fact $e^{-\eta k}$

$\eta \sim \frac{1}{\Lambda}$

$$\langle 0 | \psi_R^{+}(t,x) \, \psi_R(0,0) | 0 \rangle = \int_0^{\infty} \frac{dk}{2\pi} \, e^{-ik(t-x-i\eta)}$$

$$= \frac{1}{2\pi i(t-x-i\eta)} = \langle 0 | \psi_R(t,x) \psi_R^{+}(0,0) | 0 \rangle$$

- get similar results using Λ cut-off for $\Lambda|t-x| \gg 1$

but this is more convenient

similarly $\langle 0|\psi_L^+(t,x)\,\psi_L(0,0)|0\rangle$

$$= \frac{1}{2\pi i(t+x-i\eta)} = \langle 0|\psi_L(t,x)\,\psi_L^+(0,0)|0\rangle$$

[just take $x \to -x$]

- consider normal ordered operators where we move all creation ops a^+, b^+ to left of all annihilation ops a, b

$$J_R(t,x) \equiv :\psi_R^+(t,x)\,\psi_R(t,x):$$
$$= \psi_R^+\psi_R - \langle 0|\psi_R^+\psi_R|0\rangle$$

then $\langle 0|J_R(t,x)\,J_R(0,0)|0\rangle$

$$= \langle 0|:\psi_R^+(t,x)\,\psi_R(t,x):\;:\psi_R^+(0,0)\,\psi_R(0,0):|0\rangle$$

$$= \langle 0|\psi_R^+(t,x)\,\psi_R(0,0)|0\rangle\,\langle 0|\psi_R(t,x)\,\psi_R^+(0,0)|0\rangle$$

[other contraction cancelled by normal ordering]

$$= -\left[\frac{1}{2\pi(t-x-i\eta)}\right]^2 = \frac{1}{4\pi}\langle 0|\partial_-\varphi(t,x)\,\partial_-\varphi(0,0)|0\rangle$$

- first hint of bosonization!

(19)

Could we identify $J_R = :\psi_R^\dagger \psi_R:$
with $\pm \frac{1}{\sqrt{4\pi}} \partial_- \varphi$?

- actually, the free boson Hamiltonian
is quadratic in $\partial_\pm \varphi$

$H = \frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}(\partial_x \varphi)^2$ [Hamiltonian density]

$= \frac{1}{4}(\partial_- \varphi)^2 + \frac{1}{4}(\partial_+ \varphi)^2$

$= H_R + H_L$

- free fermion Hamiltonian is

$H = -i\, \psi_R^\dagger \partial_x \psi_R + i\, \psi_L^\dagger \partial_x \psi_L$

$\simeq \pi J_R^2 + \pi J_L^2$??

- amazingly this is true and is the
key to bosonization

Naively $J_R(x)^2 \sim \psi_R^\dagger(x)\, \psi_R^\dagger(x)\, \psi_R(x)\, \psi_R(x)$
is zero due to fermi statistics

$\psi_R(x)^2 = 0$

- but we need to be careful about operator order

- this is very singular so lets consider $\lim_{\epsilon \to 0} J_R(x) J_R(x+\epsilon)$

- this seems natural in a CMP context where ϵ could be a lattice constant

$J_R(x) J_R(x+\epsilon) = :\psi_R^\dagger(x)\psi_R(x)::\psi_R^\dagger(x+\epsilon)\psi_R(x+\epsilon):$

lets ~~rught~~ write this as $:\psi_R^\dagger(x)\psi_R(x)\psi_R^\dagger(x+\epsilon)\psi_R(x+\epsilon):$

+ corrections

- corrections arise from bringing creation operator part of $\psi_R^\dagger(x+\epsilon)$ to left of annihilation op part of $\psi_R(x)$, then bringing creation operator part of $\psi_R(x+\epsilon)$ to left of annihilation op part of $\psi_R^\dagger(x)$

- the completely normal ordered product vanishes as $\epsilon \to 0$

because creation ops are next to creation ops and annihilation next to annihilation

- the correction from bringing creation part of $\psi_R^+(x+\epsilon)$ to left of annihilation part of $\psi_R(x)$ is $\langle 0|\psi_R(x)\psi_R^+(x+\epsilon)|0\rangle$

$\times :\psi_R^+(x)\psi_R(x+\epsilon):$ etc

- altogether we have

$$:J_R(x)J_R(x+\epsilon): = :\psi_R^+(x)\psi_R(x)\psi_R^+(x+\epsilon)\psi_R(x+\epsilon):$$

$$+ :\psi_R^+(x)\psi_R(x+\epsilon): \langle 0|\psi_R(x)\psi_R^+(x+\epsilon)|0\rangle$$

$$+ \cancel{:\psi_R^+(x)\psi_R(x+\epsilon):} :\psi_R^+(x+\epsilon)\psi_R(x): \langle 0|\psi_R(x+\epsilon)\psi_R^+(x)|0\rangle$$

$$+ \langle 0|\psi_R^+(x)\psi_R(x+\epsilon)|0\rangle \langle 0|\psi_R(x)\psi_R^+(x+\epsilon)|0\rangle$$

- note no minus signs due to order written

$$J_R(x)J_R(x+\epsilon) = :J_R J_R:$$

$$\frac{1}{2\pi i\epsilon}\left[:\psi_R^+(x)\psi_R(x+\epsilon): - :\psi_R^+(x+\epsilon)\psi_R(x):\right]$$

$$- \left(\frac{1}{2\pi\epsilon}\right)^2$$

as $\epsilon \to 0$ we get a derivative

$$\lim_{\epsilon\to 0}\left[J_R(x) J_R(x+\epsilon) + \left(\frac{1}{2\pi\epsilon}\right)^2 \right]$$

$$= \frac{1}{2\pi i}\left[:\psi_R^\dagger \frac{d}{dx}\psi_R: - :\frac{d\psi_R^\dagger}{dx}\psi_R: \right]$$

so $H_R = \pi J_R^2$ as needed! [up to c-number]

similarly $J_L(x) J_L(x+\epsilon) = :J_L(x) J_L(x+\epsilon):$

$$- \frac{1}{2\pi i}\left[:\psi_L^\dagger(x)\psi_L(x+\epsilon): - :\psi_L^\dagger(x+\epsilon)\psi_L(x): \right]$$

$$- \left(\frac{1}{2\pi\epsilon}\right)^2$$

- extra minus sign here because

$$\langle 0|\psi_L(x)\psi_L^\dagger(x+\epsilon)|0\rangle = -\frac{1}{2\pi i\epsilon} = -\langle 0|\psi_R(x)\psi_R^\dagger(x+\epsilon)|0\rangle$$

so $\lim_{\epsilon\to 0}\left[J_L(x) J_L(x+\epsilon) + \left(\frac{1}{2\pi\epsilon}\right)^2 \right] = \frac{1}{\pi} H_L$

$H_L = \pi J_L^2$ [drop c-numbers]

& consistent with $J_R = \frac{1}{\sqrt{4\pi}}\partial_-\phi$

$$J_L = \frac{-1}{\sqrt{4\pi}}\partial_+\phi$$

(197)

- sign convention is arbitrary - just fixes signs of φ_R and φ_L

- commutators of these ~~bosons~~ sets of operators also correspond

$[J_R(x), J_R(y)]$ (at equal times) is clearly zero except at $x \to y$

- we can use our previous expansion in normal ordered quantities - no contribution from operator parts

$$[J_R(x), J_R(y)] = \langle 0| [J_R(x), J_R(y)] |0\rangle$$

$$= -\left[\frac{1}{2\pi(x-y+i\eta)}\right]^2 + \left[\frac{1}{2\pi(y-x+i\eta)}\right]^2$$

N.B. - only $i\eta$ term distinguishes 2 terms

- as I showed last lecture, this is

$$\frac{d}{dx}\left[\frac{1}{(2\pi)^2(x-y+i\eta)} - \frac{1}{(2\pi)^2(x-y-i\eta)}\right]$$

$$= -\frac{i}{2\pi}\frac{d}{dx}\delta(x-y)$$

this is $\frac{1}{4\pi}\left[\partial_-\varphi(x), \partial_-\varphi(y)\right]$

so $J_R(x)$ and $\frac{1}{\sqrt{4\pi}}\partial_-\varphi(x)$

have same commutator and same (quadratic) Hamiltonian

- therefore the models are equivalent

- i.e. commutation relations determine mode expansion - both are harmonic oscillator models - particle-hole pair ⇔ boson [PSS]

$J_R(x)$, $J_L(x)$ are charge densities of left and right movers

$$\int dx\, J_R(x) = \int_0^{\Lambda} \frac{dk}{2\pi}\left[a_k^\dagger a_k - b_k^\dagger b_k\right]$$

$$\int dx\, J_L(x) = \int_{-\Lambda}^{0} \frac{dk}{2\pi}\left[a_k^\dagger a_k - b_k^\dagger b_k\right]$$

$\rho = J_L + J_R$ is charge density

$J = J_R - J_L$ = current density

[in units where $V_F = 1$ - recall all

excitations travel at same speed so charge and density are simply related

- very importantly, we can extend bosonization dictionary to express $\psi_R(x)$, $\psi_L(x)$ in terms of $\varphi_R(x)$, $\varphi_L(x)$

- we can determine correct expression by considering commutator:

$$[J_R(x), \psi_R(y)]$$

$$= [\psi_R^+(x)\psi_R(x), \psi_R(y)]$$

$$= -\{\psi_R(y), \psi_R^+(x)\}\psi_R(x)$$

$$= -\delta(x-y)\,\psi_R(x)$$

- can we find an operator, written in terms of $\varphi_R(x)$, which has this commutator with $\frac{1}{\sqrt{4\pi}}\partial_-\varphi_R(x)$?

to find it, consider commutator of

$[\partial_- \varphi_R(x), \varphi_R(y)]$

$= [\partial_- \varphi_R(x), \varphi_R(y) + \varphi_L(y)]$

$= [\partial_t \varphi - \partial_x \varphi, \varphi]$

$= -i\,\delta(x-y)$

- can we think of some function $f(\varphi_R)$

$\frac{1}{\sqrt{4\pi}}[\partial_- \varphi_R(x), f(\varphi_R(y))]$

$= -\delta(x-y)\, f[\varphi_R(x)]$?

$\frac{1}{\sqrt{4\pi}}[\partial_- \varphi_R(x), \exp[-i\sqrt{4\pi}\,\varphi_R(y)]]$

$= \frac{1}{\sqrt{4\pi}} \sum_{n=0}^{\infty} \frac{(-i\sqrt{4\pi})^n}{n!} [\partial_- \varphi_R(x), (\varphi_R(y))^n]$

$= -\frac{i\,\delta(x-y)}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{(-i\sqrt{4\pi})^n}{n!} \cdot n\,[\varphi_R(y)]^{n-1}$

$= -\delta(x-y)\exp[-i\sqrt{4\pi}\,\varphi_R(x)]$

(2 a)

- so we conclude that $\psi_R(x) \propto e^{-i\sqrt{4\pi}\,\varphi_R(x)}$

- constant of proportionality not yet determined

- let's check Green's function -

$$\langle e^{i\sqrt{4\pi}\,\varphi_R(t,x)}\, e^{-i\sqrt{4\pi}\,\varphi_R(0,0)}\rangle$$

$$\propto \langle \psi_R^+(t,x)\, \psi_R(0,0)\rangle = \frac{1}{2\pi i(t-x-i\eta)} \; ?$$

- a convenient way of calculating this is to first calculate

$$\langle e^{i\beta\varphi(t,x)}\, e^{-i\beta\varphi(0,0)}\rangle$$

(for arbitrary β) then observe it must factorize

$$\propto \langle e^{i\beta\varphi_R(t,x)}\, e^{-i\beta\varphi_R(0,0)}\rangle$$

$$\langle e^{i\beta\varphi_L(t,x)}\, e^{-i\beta\varphi_L(0,0)}\rangle$$

$$= F(t-x)\cdot F(t+x)$$

- we can then extract $F(t-x)$

- first do calculation for ~~real~~ imaginary time,

$$\langle e^{i\beta\varphi(\tau,x)} e^{-i\beta\varphi(0,0)}\rangle$$

$$= \frac{1}{Z}\int[d\varphi]\, e^{-\frac{1}{2}\int d^2x\left[\left(\frac{\partial\varphi}{\partial\tau}\right)^2+\left(\frac{\partial\varphi}{\partial x}\right)^2\right] + i\beta\varphi(\tau,x) - i\beta\varphi(0,0)}$$

where $Z = \int[d\varphi]\, e^{-S_0}$

more generally, consider

$$\langle e^{i\int d^2x\, \varphi(\tau,x)\, f(\tau,x)}\rangle$$

for arbitrary function $f(\tau,x)$

$$= \langle e^{i\int \frac{d\omega dk}{(2\pi)^2}\, \tilde{\varphi}(\omega,k)\, \tilde{f}(-\omega,-k)}\rangle$$

where $\tilde{\varphi}$, $\tilde{f}$ are Fourier transforms

$$S = \frac{1}{2}\int \frac{d\omega\, dk}{(2\pi)^2}\, |\tilde{\varphi}(\omega,k)|^2\, (\omega^2+k^2)$$

- get an independent Gaussian integral

for each Fourier mode -

use $$\frac{\int dx\, e^{-\frac{a}{2}x^2 + ibx}}{\int dx\, e^{-\frac{a}{2}x^2}} = e^{-\frac{b^2}{2a}}$$

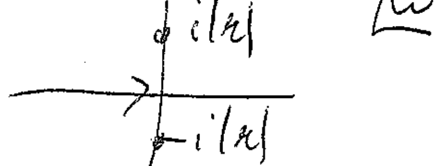
$$\left\langle e^{i\int \tilde{\varphi}\tilde{f}\,\frac{d\omega\, dk}{(2\pi)^2}}\right\rangle = e^{-\frac{1}{2}\int \frac{dk\, d\omega}{(2\pi)^2}\,\frac{|\tilde{f}(\omega,k)|^2}{\omega^2+k^2}}$$

Mar. 20

$$= \exp\left[-\frac{1}{2}\int d\tau\, dx\, d\tau'\, dx'\; f(\tau,x)\, G(\tau-\tau', x-x')\, f(\tau',x')\right]$$

where $G(\tau,x) = \int \frac{d\omega\, dk}{(2\pi)^2}\,\frac{e^{i(\omega\tau+kx)}}{\omega^2+k^2}$

- this is Green's function for φ that we calculated last lecture
- this can be seen by doing ω-integral by contour method



$$G(\tau,x) = \frac{1}{4\pi}\int \frac{dk}{|k|}\, e^{-|k||\tau| + ikx}$$

- as we saw before, this is infrared divergent, but we will see that only $G(\tau,x) - G(0,0)$ is needed

~~we~~ to calculate $\left\langle e^{i\beta[\varphi(\tau,x)-\varphi(0,0)]}\right\rangle$

we chose $f(\tau',x') = \beta\,[\delta(\tau'-\tau)\,\delta(x'-x) - \delta(\tau')\,\delta(x')]$

so $\left\langle e^{i\beta[\varphi(\tau,x)-\varphi(0,0)]}\right\rangle$

$$= e^{\beta^2[G(\tau,x)-G(0,0)]}$$

$$= e^{-\frac{\beta^2}{4\pi}\ln[\Lambda^2(\tau^2+x^2)]}$$

[for $\Lambda^2(\tau^2+x^2) \gg 1$ — Λ is cut-off]

$$= \left[\frac{1}{\Lambda^2(\tau^2+x^2)}\right]^{\beta^2/4\pi}$$

- so $\left\langle e^{i\beta\varphi_R(\tau,x)}\, e^{-i\beta\varphi_L(\tau,x)}\right\rangle$

$$= \left[\frac{1}{\Lambda(\tau-ix)}\right]^{\beta^2/4\pi}$$

- we had $\beta = \sqrt{4\pi}$

$$\rightarrow \frac{1}{\Lambda(\tau-ix)}$$

- this is correct answer

$\rightarrow \dfrac{1}{i\Lambda(t-x)}$ for real time

- similarly $\psi_L \propto e^{i\sqrt{4\pi}\,\varphi_L}$

[note opposite sign in exponent due to sign change for ~~commutation~~ current:

$J_L = +\frac{1}{\sqrt{4\pi}}\partial_t\varphi$]

- could these formulas be consistent with anti commutation relations for fermion fields?

check $\{\psi_R(x), \psi_R(y)\} = 0$

with $\psi_R(x) \propto e^{-i\sqrt{4\pi}\,\varphi_R(x)}$

- what is $[\varphi_R(x), \varphi_R(y)]$?

- it must be zero when $x=y$, but not zero in general since we showed:

$$[\partial_- \varphi(\cancel{t},x), \partial_- \varphi(t,y)]$$

$$= -2i \frac{d}{dx} \delta(x-y)$$

$$= 4[\partial_x \varphi_R, \partial_y \varphi_R]$$

$$[\varphi_R(x), \partial_y \varphi_R(y)] = -\frac{i}{2} \delta(x-y)$$

$$\Rightarrow [\varphi_R(x), \varphi_R(y)] = \frac{i}{4} \operatorname{sgn}(x-y)$$

$$\Rightarrow \psi_R(x)\psi_R(y) \neq \psi_R(y)\psi_R(x)$$

instead we get a minus sign-check!
[~~HW~~ PS 5]