Irrelevant Operator Corrections to Low Energy Fixed Point

- we may think of $\Lambda \to \infty$ (in a sense) as $D \to 0$ fixed point in Kondo problem
- low energy behaviour of the $\Lambda = \infty$ model is just free fermions with
  (1) $\pi/2$ phase shift in s-wave channel
  (2) $\frac{1}{2}$ impurity spin
- this gives most physical behaviour in zero energy (∞ length) limit
- we can go further and calculate leading corrections to this behaviour when $T \ll T_K$
- this is determined (in any model) by the leading irrelevant operator at low energy fixed point

- we can approach this using the "anything not forbidden by symmetry will occur" approach
- enumerate possible interactions in S_{eff} as $D \to \infty$
- they should:
  (1) be local - occurring only at $x = 0$
    - can expand in derivatives there
  (2) not involve impurity spin - screened
  - eliminated from low energy theory
  - a bit like strongly coupled spins in PS3 last question
  (3) respect $SU(2)$ spin-rotation symmetry
  (4) respect particle hole symmetry if microscopic model had it (assume for simplicity)

allowed

- 2 Λ operators scaling as $\frac{1}{L}$:

$$\lambda_1 \sum_{\alpha} \left[\psi^\dagger_\alpha \frac{d}{dx} \psi_\alpha - \frac{d}{dx} \psi^\dagger_\alpha \psi_\alpha \right] (x=0)$$

$$+ \lambda_2 \sum_{\alpha,\beta} \left[\psi^\dagger_\alpha \psi_\alpha - \langle \psi^\dagger_\alpha \psi_\alpha \rangle \right] \left[\psi^\dagger_\beta \psi_\beta - \langle \psi^\dagger_\beta \psi_\beta \rangle \right] (x=0)$$

$\psi(\tau, s) \to s^{-1/2} \psi(\tau, s)$ after Fourier transform

$x \to s x$, $\tau \to s \tau$

$$\int d\tau \left[\bar{\psi}^\dagger \frac{d}{dx} \psi \right]_{(0)} \to s \cdot s^{-1/2} \cdot s^{-1/2} \cdot \frac{1}{s} \propto \frac{1}{s}$$

similarly $\int d\tau \left(\bar{\psi}^\dagger \psi \right)^2$

- normalize $\psi(x)$ as $\{\psi(x), \psi^\dagger(y)\} = 2\pi \delta(x-y)$

$\Rightarrow$ [illegible]$^{1/2}$ $\psi \sim (\text{length})^{-1/2}$

as $\lambda_i^* \sim$ energy $\cdot$ (length)^2

- we can estimate λ_i from dimensional analysis

- what is <u>largest</u> quantity we can construct from available parameters with right dimension?

$$\lambda_i \sim \frac{v_F^2}{T_K}$$

- actually an argument can be given to determine ratio $\frac{\lambda_1}{\lambda_2}$ exactly
- based on "spin-charge separation" in ~~Luttinger liquid~~ Dirac fermion model
[Affleck - Nucl. Phys. B336, 517, 1990.]
Nozieres 1975
- so there is only one unknown parameter of $O(1)$: λ_1 in units of V_F^2/T_K
- we can predict many low energy properties in terms of it [specific heat, magnetic susceptibility, resistance] and it drops out of ratios made from these quantities
- consider contribution of λ_1 to resistivity
- gives k-dependence to phase shift

$$\delta = \frac{\pi}{2} + \alpha (k - k_F) \frac{V_F}{T_K}, \quad \alpha \sim O(1)$$

$$\frac{1}{T}\alpha \int m^2 d\tau \sim \omega_s^2 \alpha (k-k_F)\frac{V_F}{T_K} \sim \frac{1}{2}\left(1 + i\omega 2\alpha \frac{(k-k_F)V_F}{T_K}\right)$$

$$\sim 1 - \alpha^2 (k-k_F)^2 \frac{V_F^2}{T_K^2}$$

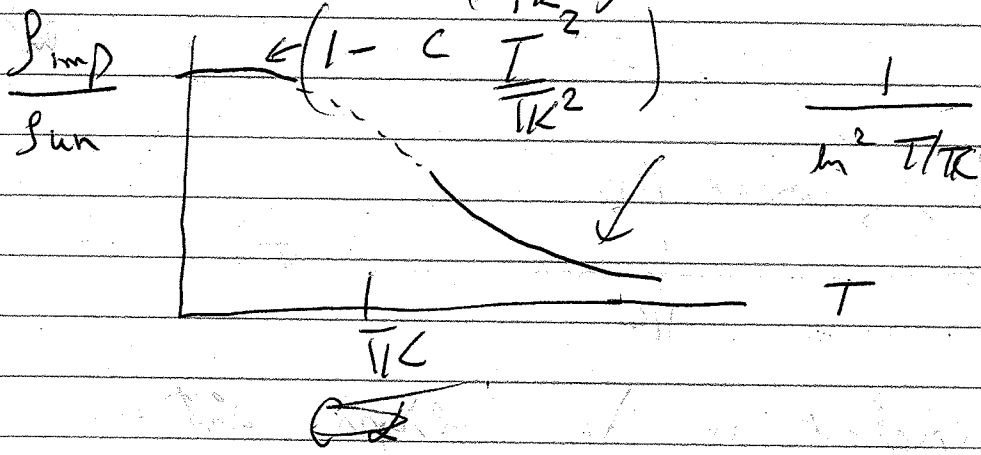
$$\int \delta\beta \sim \left(\frac{V_F}{T_K}\right)^2 \int d^3k\, (k-k_F)^2 \frac{d n_F}{d\epsilon_k}$$

$$V_F(k-k_F) \sim V_F \frac{\varepsilon}{V_F k_F} \varepsilon$$

$$\int d\varepsilon\, \varepsilon^2 \frac{d n}{d\varepsilon} \sim T^2$$

– also a similar contribution from k

$$\Rightarrow \int \beta \alpha \left(\frac{T}{T_K}\right)^2$$



$C \alpha \lambda_1^2$ (in units of V_F^2/T_K)

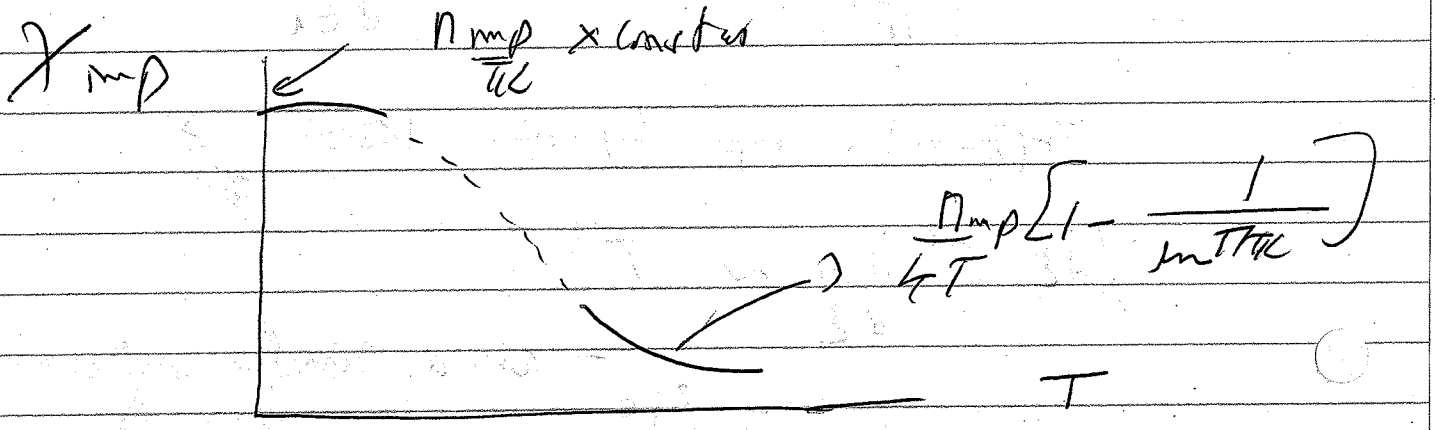
- Can also calculate magnetic susceptibility

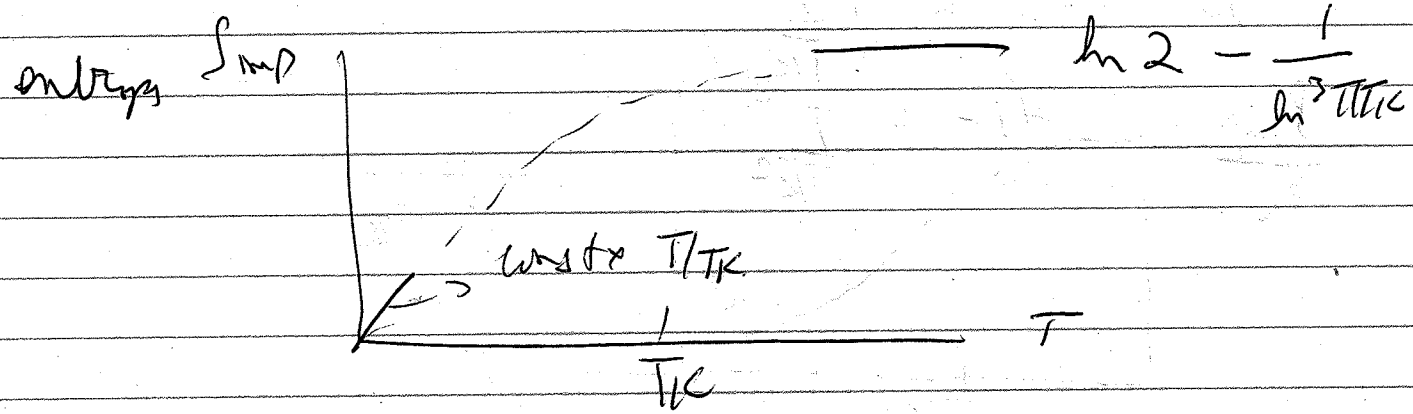
- ~~gets a~~ vanishes at $\lambda_1 = 0$ since that is

by expanding in fixed point H_c

$$H_{eff} = iv_F \int_{-\infty}^{0} dx\ \psi^\dagger \frac{d}{dx} \psi$$

$$\chi_{imp} = \frac{n_{imp}}{T_K} \times \text{const} \quad \left[1^{st} \text{ order in } \lambda_1, \lambda_2\right]$$





- unknown const in λ_1 drops out in

Wilson ratio $\quad \frac{\delta\chi/\chi}{\delta c/c} = 2.$

- many other quantities can be calculated

in a similar way - RG improved perturbation theory in λ [Kondo coupling] at high energies, - lowest order perturbation theory in λ_i [leading irrelevant operator] at low energies -

$M(H)$, $S(H)$, density oscillations, Knight shift, ...

- obtain good agreement with other theoretical techniques at high and low energies - Bethe ansatz & Numerical Renormalization Group - show graphs
- so brilliant guess about low energy fixed point was correct
- only ~~cross~~ cross over regime, $T \sim T_K$, is difficult & requires more numerically intensive methods

- agreement with experiments is good at qualitative level but more realistic model is usually needed [larger impurity spin magnitude ($S > \frac{1}{2}$), charge fluctuation of magnetic impurity atom (anderson model), finite impurity concentrations, realistic band structure, non-s-function Kondo interaction, other, non-magnetic impurities, ...]

- nonetheless Kondo model is a nice illustration of the RG at work

- both high energy and low energy behaviour are accessible with minimal numerical work

Interacting fermions in 1 Dimension

- experimental applications: carbon nano-tubes, metallic nano-wires, gated semi-conductor nano-wires, nano-wires self-assembled on metallic surfaces, organic conductors, quasi-1D magnetic insulators [by Jordan-Wigner transformation]
- Consider simplest case - single band of spinless fermions with short-range (screened) Coulomb interactions
- RG is much simpler in 1D than 2 & 3D
- behaviour is also more exotic [non-Fermi liquid]
- I will follow treatment in Shankar Chp IV quite closely - please read it!
- we will apply RG and focus on universal low energy properties

- these are largely independent of microscopic model
- but we could consider a tight binding model

$$H = \sum_j \left[-t\left(\psi_j^\dagger \psi_{j+1} + h.c.\right) + V \hat{n}_j \hat{n}_{j+1} \right]$$

where $\hat{n}_j \equiv \psi_j^\dagger \psi_j$

- NB- Kondo interaction was "zero dimensional"
- now we have "graduated" to $D = 1$
- nearest neighbour interaction V is shortest-ranged possible for spinless fermions (why?)
- for $V = 0$, or for constructing perturbation theory in V we Fourier transform ψ_j:

$$\psi_j = \sqrt{a} \int_{-\pi/a}^{\pi/a} \frac{dK}{2\pi} \, e^{iKja} \psi(K)$$

- same notation as Shankar, but I keep lattice spacing a
- note ∞ length limit, K continuous

- K- integral over Brillouin zone only

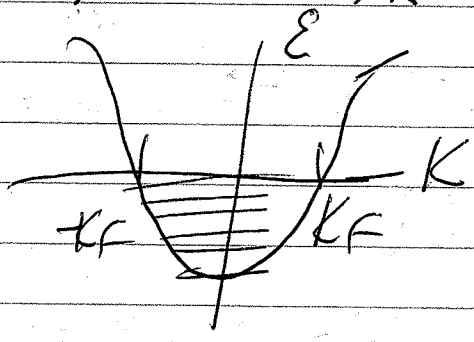
$\psi(k)$ normalized as in Shankar: $\{\psi(k), \psi^\dagger(k')\}$

$= 2\pi\, \delta(k-k')$

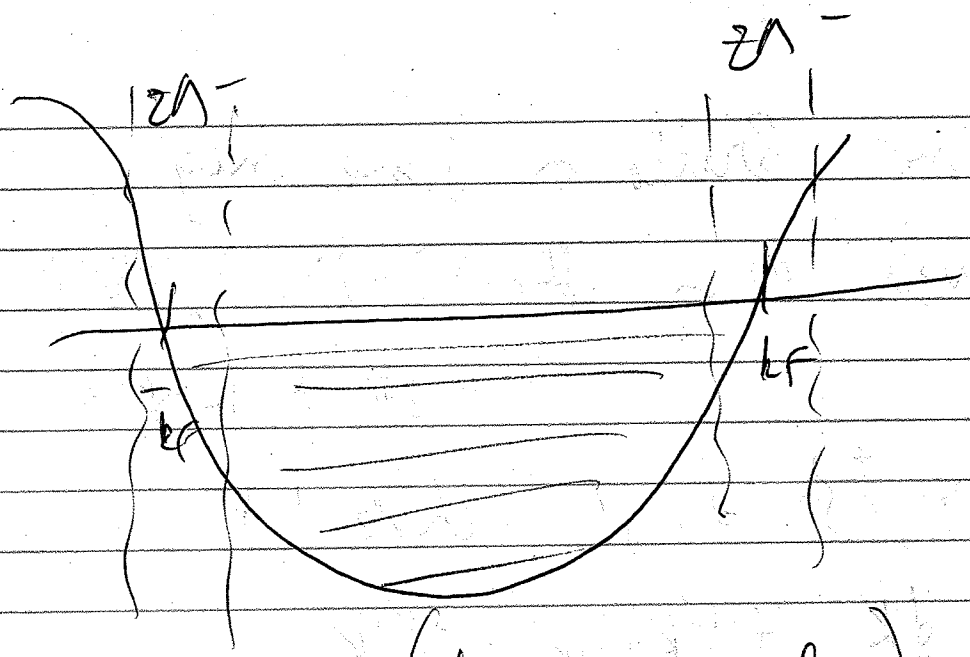
$\Rightarrow \{\psi_j, \psi_{j'}^\dagger\} = \delta_{j,j'}$ (check!)

$$H_0 = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi}\, \psi^\dagger(k)\psi(k)\, \varepsilon(k)$$

$$\varepsilon(k) = -2t \cos ka - \mu$$



- only 2 fermi points, $\pm k_F$, not a fermi surface - makes things much simpler
- we will integrate out fourier modes far from fermi points in RG
- after some initial stages of RG, remaining states are 2 narrow bands near $\pm k_F$ - width Λ



- low energy ($|\varepsilon(k)|$ small) modes

have wave-vector $K = k_F + k$ (right-movers)

or $K = -k_F + k$ (left-movers)

with $|k| \ll k_F \sim \pi/2a$

let $\psi_R(x) \equiv \psi(k_F + k)$, $\psi_L(x) = \psi(-k_F + k)$

- right and left-moving fields

$$\psi_j \simeq e^{ik_F ja} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} e^{ikja} \psi_R(k) + e^{-ik_F ja} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} e^{ikja} \psi_L(k)$$

$$= e^{ik_F ja} \psi_R(ja) + e^{-ik_F ja} \psi_L(ja)$$

$\psi_R(x)$, $\psi_L(x)$ vary slowly on lattice scale.

[N.B Shankar uses non-standard notation $\psi_L(k) \to \psi_L(-k)$]

for small Λ, we can linearize dispersion relation: $\mathcal{E}(K_F + k) \simeq v_F k$

$$\mathcal{E}(-K_F + k) \simeq -v_F k.$$

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$v_F = 2t \sin k_F$ in tight-binding model

$$H_0 = v_F \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} k \left[\psi_R^\dagger(k) \psi_R(k) - \psi_L^\dagger(k) \psi_L(k) \right]$$

$$= -i v_F \int dx \left[\psi_R^\dagger \frac{d}{dx} \psi_R - \psi_L^\dagger \frac{d}{dx} \psi_L \right]$$

[$\psi_L(x), \psi_R(x)$ slowly varying on lattice scale since $\Lambda \ll \pi/a$]

action $S = i \int dt\, dx \left[\psi_R^\dagger (\partial_t - i v_F \partial_x) \psi_R + \psi_L^\dagger (\partial_t + v_F \partial_x) \psi_L \right]$

- this is Lorentz invariant [with speed of light, c, replaced by v_F]

- "boost": $(v_F t \pm x) \to e^{\pm \alpha} (v_F t \pm x)$

$$\begin{pmatrix} v_F t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} v_F t \\ x \end{pmatrix}$$

($\tanh\alpha = \frac{u}{v_F}$ where u is velocity of moving reference frame)

action is invariant under boost if

we also transform fields

$$\psi_R' = e^{-\frac{\alpha}{2}}\psi_R, \quad \psi_L' = e^{+\frac{\alpha}{2}}\psi_L$$
$$\psi_R'^{+} = e^{-\alpha/2}\psi_R^{+}, \quad \psi_L'^{+} = e^{+\alpha/2}\psi_L^{+}$$

- this is relativistic Dirac fermion model in (1+1) dimension

- imaginary time action, appearing in Feynman path integral, is

$$S_0 = \int_{-\infty}^{\infty}\frac{dw}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\left[\psi_L^{+}(iw + vFk)\psi_L + \psi_R^{+}(iw - vFk)\psi_R\right]$$

- w is Fourier transform of imaginary time

$= \frac{2\pi}{\beta}(n+\frac{1}{2})$ at finite temperature but we will mainly work at $T=0$ for simplicity

Including interactions

$$S_{INT} = \left(\prod_{i=1}^{4} \int \frac{dK_i}{2\pi} \frac{d\omega_i}{2\pi} \right) 2\pi \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

$$\delta(K_1 + K_2 - K_3 - K_4)\ \bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1)\ U(\overset{4,3,2,1}{1,2,3,4})$$

- Shankar's notation $(1) \to (K_1, \omega_1)$ etc
- time & space translation invariance imply frequency & momentum are conserved

$U(1,2,3,4)$ means $U(K_1, K_2, K_3, K_4)$

- in narrow band case each K_i is close to $\pm K_F$

$$\int dK \to \int_{-K_F-\Lambda}^{-K_F+\Lambda} dK + \int_{K_F-\Lambda}^{K_F+\Lambda} dK$$

- actually momentum only needs be conserved modulo $2\pi/a$ because spatial δ-function comes from $\frac{a}{2\pi} \sum_{j=-\infty}^{\infty} e^{i(K_1 + K_2 - K_3 - K_4)aj}$

$$= \sum_n \delta(K_1 + K_2 - K_3 - K_4 - 2\pi n/a)$$

"Umklapp"

but $K_1 + K_2 - K_3 - K_4$ can only be close to

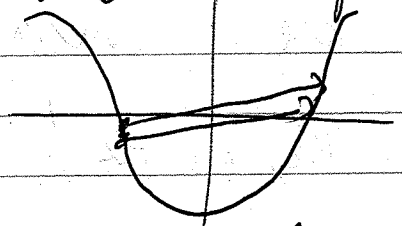
$0, \pm 2K_F$ or $\pm 4K_F$

$2K_F = 2\pi/a$ implies a full band - not interesting

$4K_F = 2\pi/a$, $K_F = \pi/2a$ - this is an

interesting special case - half-filling

- corresponding interaction processes are called Umklapp



$\psi_R^+ \psi_R^+ \psi_L \psi_L$

- 2 particles jump from left to right

Fermi point: $-\frac{\pi}{2a} \to \frac{\pi}{2a}$ at $\frac{1}{2}$-filling

- since $\psi(K_1)\psi(K_2) = -\psi(K_2)\psi(K_1)$

(or $\psi(K_1, \omega_1)\psi(K_2, \omega_2) = -\psi(K_2, \omega_2)\psi(K_1, \omega_1)$)

we can replace $U(K_4, K_3, K_2, K_1)$ by

$\frac{1}{2}\left[U(K_4, K_3, K_2, K_1) - U(K_4, K_3, K_1, K_2) \right]$

$\to \frac{1}{4}\left[U(K_4, K_3, K_2, K_1) - U(K_4, K_3, K_1, K_2) - U(K_3, K_4, K_2, K_1) + U(K_3, K_4, K_1, K_2) \right]$

- So u can be taken to be anti-symmetric in $(1,2)$ and $(3,4)$

$$u(4,3,2,1) = -u(4,3,1,2) = -u(3,4,2,1) = u(3,4,1,2)$$

- u will generally be a smooth function [see Shankar for nearest neighbour example]

- So, once we have reduced band-width to $\Lambda \ll K_F$ we may Taylor expand u around $K_i = \pm K_F$

- only 0^{th} order terms are marginal perturbations, the others are irrelevant since $\psi \to s^{3/2}\psi'$

$$dK\, dw \to \frac{dK\, d\tilde{w}}{s^2}$$

$$\bar{\psi}\bar{\psi}\psi\psi = s^6\, \bar{\psi}'\bar{\psi}'\psi'\psi'$$

$$\prod_{i=1}^{3} \int dK_i\, dw_i \to \frac{1}{s^6} \prod_{i=1}^{3} dK_i\, d\tilde{w}_i \quad \text{[using } \delta\text{-function]}$$

- So, ignoring irrelevant terms, we can label

$U(4,3,2,1)$ by discrete labels $\pm K_F$

eg $U(K_F, -K_F, -K_F, K_F) = U_{RLLR}$ etc.

$U_{LLRR} = 0$ due to anti-symmetry

(Umklapp terms) only 1 non-zero

marginal term $U_{LRLR} = -U_{RLLR} = -U_{LRRL} = U_{RLRL} \equiv U_0 \cdot 4$

$$S_{INT} = U_0 \prod_{i=1}^{4} \int \frac{dw_i}{2\pi} \frac{dk_i}{2\pi} \delta(w_1 + w_2 - w_3 - w_4)$$

$$\delta(k_1 + k_2 - k_3 - k_4)\ \bar{\psi}_L(w_4, k_4)\, \bar{\psi}_R(w_3, k_3)\, \psi_L(w_2, k_2)\, \psi_R(w_1, k_1)$$

where now $K = \pm K_F + k$, for ψ_R or ψ_L

- for Umklapp term at $\pm$-Fermi, leading non-zero term from Taylor expanding $U(K_4, K_3, K_2, K_1)$ around Fermi points is

$$2 (k_4 - k_3)(k_2 - k_1)\, U_0\, \bar{\psi}_L(4)\, \bar{\psi}_L(3)\, \psi_R(2)\, \psi_R(1)$$

$\longrightarrow \frac{1}{\Lambda^2}$ under RG

- so we can ignore this for small enough bare coupling

but actually, for large enough bare coupling it drives a phase transition into a charge density wave phase $\Phi_1 \cdot \Phi_2 \cdot \Phi_3$

- similarly terms with more fermion fields or more powers of ω are irrelevant

So contain all marginal terms with 2 fermion fields

[we could add a $(\bar{\psi}_L \psi_L + \bar{\psi}_R \psi_R)$ term but this just shifts chemical potential - instead we always chose bare μ to cancel such terms and keep chemical potential at desired value $\langle n \rangle = \int_{-k_F}^{k_F} \frac{dk}{2\pi} = \frac{k_F}{\pi}$ is exact - Luttinger theorem

- proven non-perturbatively in $D=1$ - Yamanaka-Oshikawa-Affleck, PRL 79, 1110 (1997)]

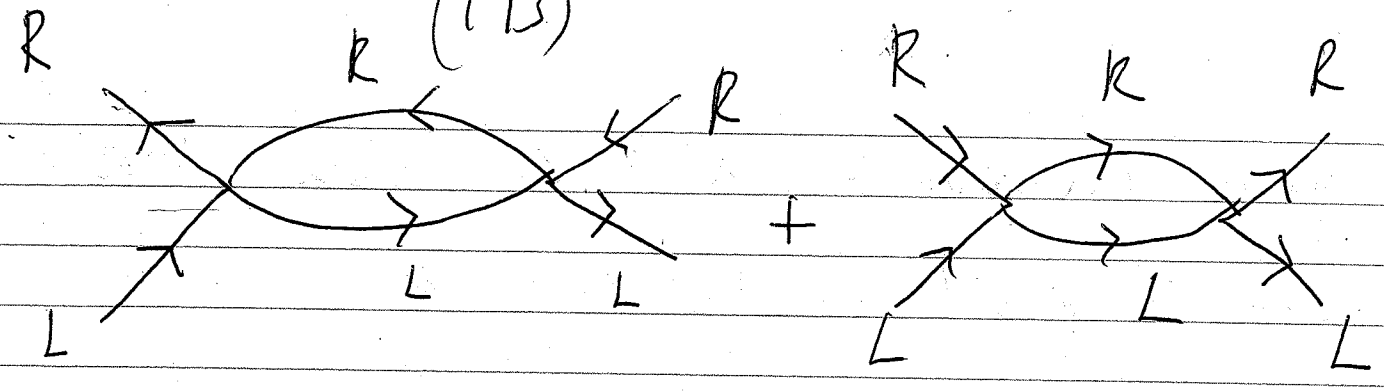
- interactions do renormalize velocity, v_F, of low energy excitations
- note that marginal interaction is Lorentz invariant since $\psi_R \to e^{-\frac{\alpha}{2}} \psi_R$, $\psi_L \to e^{\frac{\alpha}{2}} \psi_L$
- now consider RG transformation, to 2^{nd} order in U_0, in Tomonaga-Luttinger theory
- integrate out modes with $\frac{\Lambda}{s} < |k| < \Lambda$ reducing cut-off by a factor of $s > 1$
- 2 types of terms generate a correction to U

$$(\bar{\psi}_L \bar{\psi}_R \psi_L \psi_R)(\bar{\psi}_L \bar{\psi}_R \psi_L \psi_R)$$

and

$$(\bar{\psi}_L \bar{\psi}_R \psi_L \psi_R)(\bar{\psi}_L \bar{\psi}_R \psi_L \psi_R)$$

diagrammatically these are



called ZS' and BCS

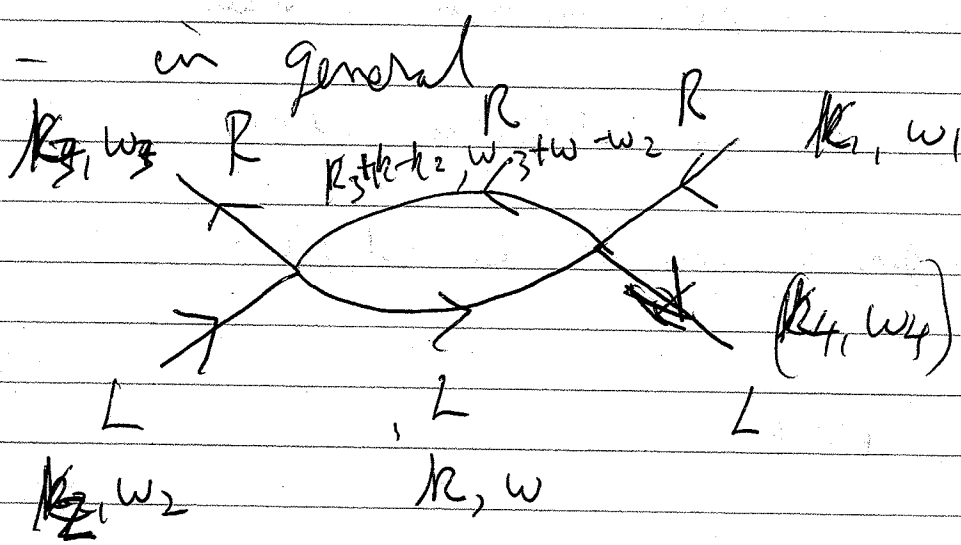
diagrams in Shankar.

- momenta on external lines must be in > range

$$\Lambda < |k| < \Lambda.$$

- external momenta are in < range

- in general

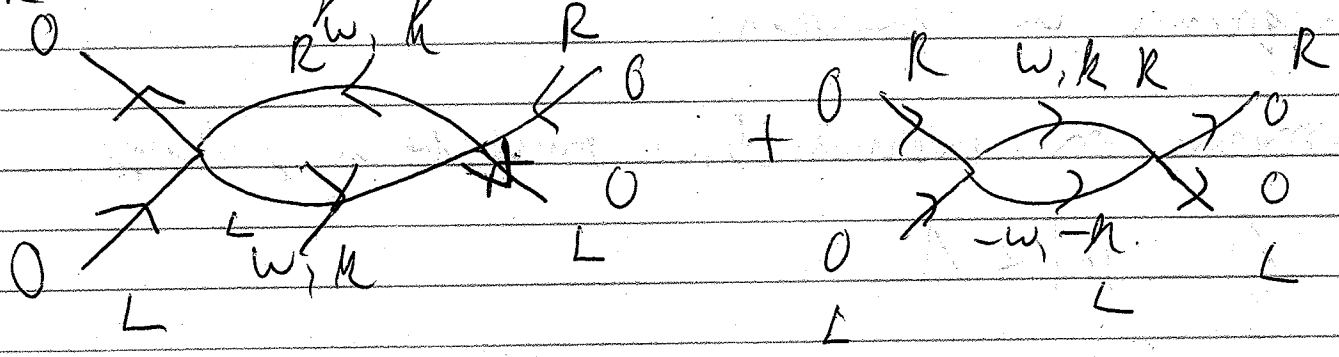


- this will give us a correction to u that depend on (K_i, ω_i)

- but we could again Taylor expand it

- gives correction to $U(v)$ of interest together with irrelevant coupling constants

- to get marginal part we can set all external frequencies and momenta to zero



- note that both ZI and BCS diagram contain 1 R internal line and 1 L internal line

- corresponding propagators can be deduced from S_0:

$$\xrightarrow{R\ w,k} = \frac{\beta}{iw - vFk}$$

$$\xrightarrow{L\ w,k} = \frac{\beta}{iw + vFk}$$

taking into account various terms minus signs we get:

$$\delta u = -u^2 \int_{-\infty}^{\infty} \frac{dw}{2\pi} \int' \frac{dk}{2\pi} \frac{1}{(iw - v_F k)(iw + v_F k)}$$

$$-u^2 \int_{-\infty}^{\infty} \frac{dw}{2\pi} \int' \frac{dk}{2\pi} \frac{1}{(iw - v_F k)(-iw - v_F k)}$$

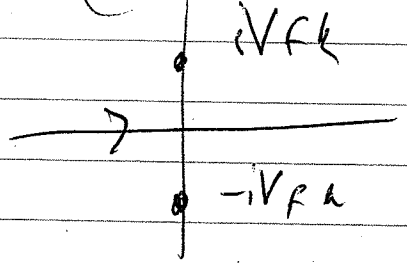
from Z_S' and BCS diagrams respectively

here $\int' k = \int_{-\Lambda}^{\Lambda_S} dk + \int_{\Lambda_S} dk.$

[factor of $\frac{1}{2!}$ from 2^{nd} order Taylor expansion of S_{INT} cancels against interchanging 2 vertices]

- please check minus signs for both terms!
- integrals over w can be done, for example, as contour integrals in complex plane.

$$I = \int_{-\infty}^{\infty} \frac{dw}{2\pi} \frac{1}{(iw - v_F k)(iw + v_F k)}$$



close in upper half-plane

$$I = \frac{i}{i\,(-2v_F|k|)} = -\frac{1}{2v_F|k|}$$

(even function of k)

$$\int^{\Lambda} \frac{dk}{|k|} = 2\int_{s}^{\Lambda} \frac{dk}{k} = 2\ln\frac{\Lambda}{s}$$

$$\delta U_{zs'} = \frac{U^2}{v_F}\ln\frac{\Lambda}{s}$$

$$\delta U_{BCS} = -\frac{U^2}{v_F}\ln\frac{\Lambda}{s}$$

- diagrams cancel!