- this energy scale is called the Kondo temperature, T_K [setting $k_B = 1$]

using $\lambda(E) = \dfrac{\lambda_0}{1 - \lambda_0 \ln \frac{D_0}{E}}$

we can estimate T_K, using $\lambda(T_K) \simeq 1$

$$1 \simeq \frac{\lambda_0}{1 - \lambda_0 \ln \frac{D_0}{T_K}} = \frac{1}{\frac{1}{\lambda_0} - \ln \frac{D}{T_K}}$$

$$\frac{1}{\lambda_0} \simeq \ln \frac{D_0}{T_K},$$

$$T_K \simeq D_0 e^{-\frac{1}{\lambda_0}} \quad [\text{up to a factor of } O(1)]$$

in this rough estimate we could take D_0 to be the original bandwidth or E_F and to be the original "bare" Kondo coupling

$\sim E_F$, $T_K \simeq E_F \exp\left[-\frac{1}{J\nu}\right]$

- exponentially small compared to E_F

- like in QCD - strong interaction coupling constant also grows with lowering E

- RG improved pert. theory works for $E \gg T_K$, but at $E \tilde{>} T_K$ we enter strong-coupling regime where pert. theory completely fails (even when improved by RG)

- it is convenient to write $\lambda(E) \simeq \frac{1}{\ln E/T_K}$

for $E \gg T_K$ - elementary bare coupling λ_0 we replaces it by T_K

- RG improved P.T. can be applied to resistivity as I will now show

- lowest order $R(T) \propto n_{imp} \lambda_0^2$ where n_{imp} = impurity concentration - NB - this is just contribution to $R(T)$ from scattering off magnetic impurities - must be added to other contributions

- now if we RG-improve calculation we

get $R(T) \propto \frac{\hbar m_p}{n e^2 T/T_K}$ for $T \gg T_K$

- this fits experimental data & numerical calculations of (Gorkov model) well at $T \gg T_K$

- NB - we cannot use this result as $T \to T_K$ where it predicts $R \to \infty$

- I will return to question of what happens at lower T, but first go thru calculation of $R(T)$ to 2^{nd} order in J in some detail - opportunity to learn some other techniques of CMP

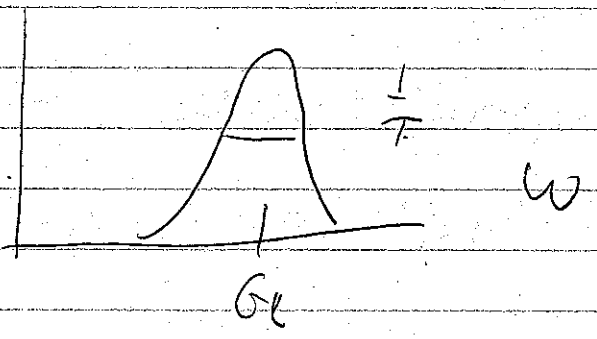
- see Hewson, Chap 2, Mahan 11.1, AGM chap 13

- the plan is to calculate (one) electron Green's function

$$G_{ret}(\vec{k}, \omega) = \frac{1}{\omega - \tilde{\varepsilon}_{\vec{k}} + \frac{i}{2\tau}}$$

- this broadens single-electron pole

- $\mathrm{Im}\, G_{ret}$



width = $\frac{1}{\tau}$, τ is time electron spends in a state of momentum $\vec{k}$ before scattering off an impurity)

- we can insert this result in Boltzmann equation result for resistivity

$$\sigma = \frac{1}{R} = \frac{2e^2}{3} v_F^2 \cdot \nu \cdot \tau$$

where ν = density of states at Fermi surface

- Phys 502? - see Ac M

- Boltzmann equation approach makes it clear that scattering rate at energies $|G_{\vec{k}}| \lesssim \tau$ contribute to resistivity

at temperature T

- using free electron dispersion relation this reduces to $\frac{1}{R} = \frac{ne^2}{m}\tau$ – classical Drude formula

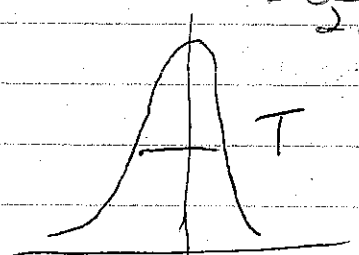
- most rigorous derivation of this formula is from Kubo formula – can express retarded GF for current operator in terms of GF for electron operator in this case

[AeM (13.25)

$$\sigma^{ab} = e^2 \int \frac{d^3k}{4\pi^3}\, \tau[\varepsilon_{\vec{k}}]\, V_{\vec{k}}^a V_{\vec{k}}^b \left(-\frac{\partial f}{\partial \varepsilon}\right)_{\varepsilon_{\vec{k}}}$$

- for isotropic dispersion relation can replace $V_{\vec{k}}^a V_{\vec{k}}^b \to \frac{1}{3} \delta^{ab} V_{\vec{k}}^2$ inside integral

$\left(V_{\vec{k}} = \nabla \varepsilon_{\vec{k}}\right)$

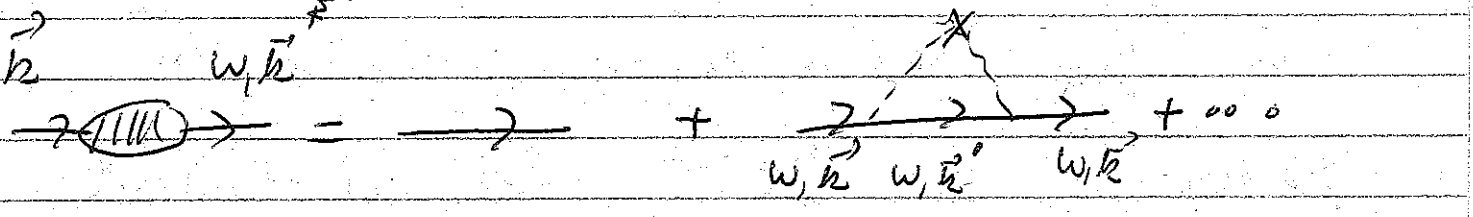


τ constant $\Rightarrow \sigma$ for $\varepsilon_k \lesssim T$

$= 2^{nd}$ half of Phys 502]

14/2

Calculation of Electron GF $-\langle \psi_{\vec{k}} \bar{\psi}_{\vec{k}'} \rangle$



- we will average over positions of dilute random array of impurities - restores translation invariance and makes G ~~a function of~~ vanish unless $\vec{k} = \vec{k}'$
- similar to our calculation of $\Sigma\lambda$ except that now we sum over impurity state and over all momenta in action, not just in strip $D' < E_{\vec{k}}^{\pm} < D$ as for RG
- NB - no $O(\lambda)$ contribution due to $\int \vec{S} = 0$
- convenient to work in position-frequency space

$$\mathcal{G}_{\alpha\beta}(\vec{r},\vec{r}';i\omega_n) = -\langle \psi_\alpha(\vec{r},\omega_n)\,\bar{\psi}_\beta(\vec{r}',\omega_n)\rangle$$

where $\langle\rangle$ includes trace over impurity spin states

$$\delta\mathcal{H} = -\frac{J^2}{2!}\,\delta$$

$$\delta_{INT} = -J\int_{-\infty}^{\infty} d\tau\ \bar{\psi}(\vec{0},\tau)\,\frac{\vec{\sigma}}{2}\,\psi(\vec{0},\tau)\cdot\vec{S}(\tau)$$

$$\delta\mathcal{H}_{\alpha\beta} = -\frac{J^2}{2!}\sum_{ab}\langle \bar{\psi}\left(\bar{\psi}\frac{\sigma^a}{2}\psi\right)\left(\bar{\psi}\frac{\sigma^b}{2}\psi\right)\psi\rangle\langle S^a S^b\rangle$$

$\langle S^a S^b\rangle = \frac{1}{4}\delta^{ab}$ — no τ-dependence.

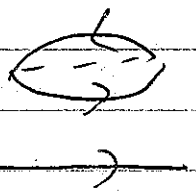
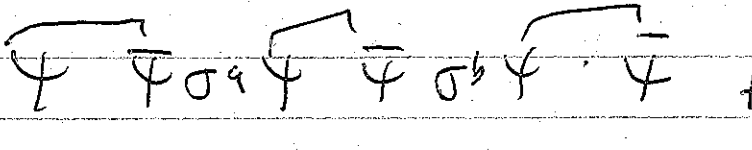


diagram corresponds to a correction to ground state energy

- only need consider



+ equivalent diagram

(sum of 2)

- ~~Contraction of electron spin indices gives~~

~~$\frac{\sigma^a}{2}\frac{\sigma^b}{2} = (\sigma^a \sigma^b)_{\alpha\beta}$~~

~~sum over multiply~~

$$\sum_{ab} \left(\frac{\sigma^a}{2}\frac{\sigma^b}{2}\right)_{\alpha\beta} \frac{1}{4}\delta^{ab} = \frac{3}{16}\delta_{\alpha\beta}$$

[using $(\sigma^a)^2 = I$]

using $\mathcal{J}^{(0)}(\vec{r}-\vec{r}', i\omega_n) = -\langle \psi(\vec{r}, \omega_n)\, \bar{\psi}(\vec{r}', \omega_n)\rangle$

we obtain

$$\sum_{\alpha\beta} \mathcal{J}(\vec{r}, \vec{r}', i\omega_n) = \frac{3}{16}\delta_{\alpha\beta} J^2$$

$$\mathcal{J}^{(0)}(\vec{r}, i\omega_n)\, \mathcal{J}^{(0)}(\vec{0}, i\omega_n)\, \mathcal{J}^{(0)}(-\vec{r}', i\omega_n)$$

- frequency is conserved since no T-dependence

erases from energy spin operators in this case.

$$\mathcal{J}^{(0)}(\vec{0}, i\omega_n) = \frac{1}{V}\sum_{\vec{k}} \frac{1}{i\omega_n - \epsilon_{\vec{k}}}$$

- return to evaluating this later

- convenient to work

$$\sum \mathcal{J}(\vec{r}, \vec{r}', i\omega_n) = \mathcal{J}^{(0)}(\vec{r}, i\omega_n)\, T(i\omega_n)\, \mathcal{J}^{(0)}(-\vec{r}', i\omega_n)$$

(suppressing $\alpha\beta$)

where $T(i\omega_n) = \frac{3}{16} J^2 \frac{1}{V} \sum_{\vec{k}} \frac{1}{i\omega_n - \xi_{\vec{k}}}$

N.B T is a function of frequency only, not position (or momentum)

- it is called the T-matrix and is a generalization of the T-matrix in ordinary potential scattering $\left(S = I + iT\right)$

- suppose impurity was not at origin, but at arbitrary location $\vec{R}$

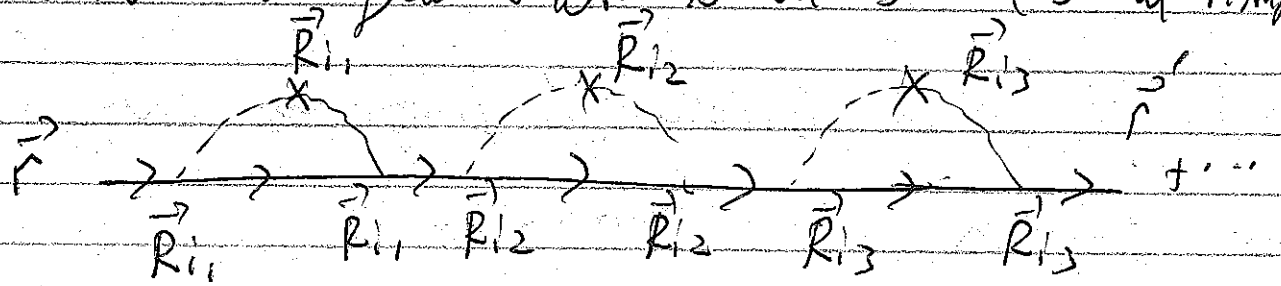
then $\mathcal{G}(\vec{r}, \vec{r}', i\omega_n) = \mathcal{G}^{(0)}(\vec{r} - \vec{R}, i\omega_n)\, T(i\omega_n)\, \mathcal{G}^{(0)}(\vec{R} - \vec{r}', i\omega_n)$

due to $\psi(\vec{r})\,\bar{\psi}(\vec{R})\;\psi(\vec{R})\,\bar{\psi}(\vec{R})\;\psi(\vec{R})\,\bar{\psi}(\vec{r}')$

- now suppose we have many impurities at locations $\vec{R}_1, \vec{R}_2, \ldots$

$$H_{INT} = J \sum_{i=1}^{N_{imp}} \psi^+(\vec{R}_i) \frac{\vec{\sigma}}{2} \psi(\vec{R}_i) \cdot \vec{S}_i$$

- dominant diagrams at small J and small n_{imp} are



$$= \mathcal{G}^{(0)}(\vec{r} - \vec{R}_{i_1}) T \mathcal{G}^{(0)}(\vec{R}_{i_1} - \vec{R}_{i_2}) T \mathcal{G}^{(0)}(\vec{R}_{i_2} - \vec{R}_{i_3}) T$$

$$\cdots T \mathcal{G}^{(0)}(\vec{R}_{i_n} - \vec{r}')$$

where $n = 0, 1, 2, 3, \ldots \infty$

- all $\mathcal{G}^{(0)}$'s and T's at same frequency, $i\omega_n$

- now sum over $i_1, i_2, i_3 \ldots i_n$ from 1 to N_{imp}

- in dilute limit we can ignore diagrams where same impurity occurs more than once in a diagram

$$\sum_{i_p=1}^{N_{imp}} \rightarrow n_{imp} \int d^3 \vec{R}_p$$

$\vec{R}_{i_p}$

since impurity locations are random and since we will also be integrating over $\vec{r}$ and $\vec{r}'$ to get resistivity

n^{th} order correction is

$$\left[n_{imp}\, T(i\omega_n)\right]^n \int d^3\vec{R}_1 \cdots d^3\vec{R}_n$$

$$g^{(0)}(\vec{r}-\vec{R}_1)\, g^{(0)}(\vec{R}_1-\vec{R}_2) \cdots g^{(0)}(\vec{R}_{n-1}-\vec{R}_n)\, g^{(0)}(\vec{R}_n-\vec{r}')$$

- NB - translation invariance is restored by averaging over impurity locations

- in $\vec{k}$ space

$$\delta(\vec{k}_1+\vec{k}_2)\left(n_{imp}\, T(i\omega_n)\right)^n \left[g^{(0)}(\vec{k}, i\omega_n)\right]^{n+1}$$

- sum series

$$g(\vec{k}, i\omega_n) = \sum_{n=0}^{\infty} \left[g^{(0)}(\vec{k}, i\omega_n)\right]^{n+1} \left[n_{imp}\, T(i\omega_n)\right]^n$$

$$= \frac{g^{(0)}}{1 - g^{(0)} n_{imp} T} = \frac{1}{g^{-1} - n_{imp} T}$$

$$= \frac{1}{i\omega_n - \xi_{\vec{k}} - n_{imp}\, T(i\omega_n)}$$

- this has the form of a self-energy

$$G = \frac{1}{i\omega_n - \vec{k}^2 - \Sigma(i\omega_n, \vec{k})}$$

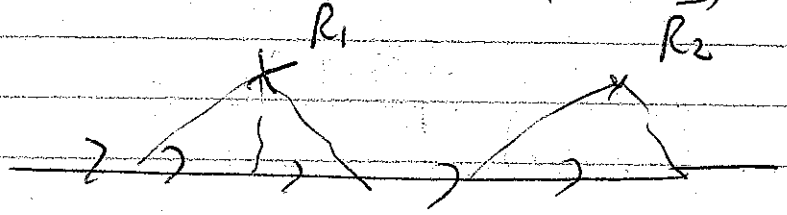
where self-energy depends on $i\omega_n$ only, not $\vec{k}$

$$\Sigma(\vec{k}, i\omega_n) = n_{imp}\, T(i\omega_n)$$

- we actually want retarded GF & define $T(\omega)$ which appears in & resistivity

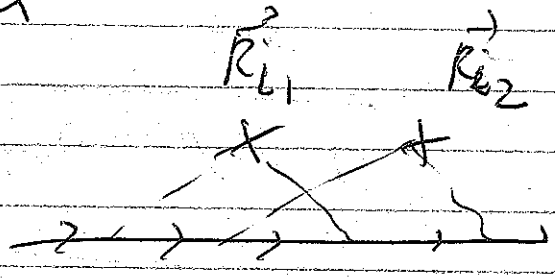
- other diagrams?

- corrections like



still give $\Sigma = n_{imp}\, T(i\omega_n)$ but give corrections to $T(i\omega_n)$ of higher order in λ

diagram like



mix different impurities - now ~~diagr~~

sums over $i_1, i_2 = 1 \cdots N_{imp}$ can't be done independently - give corrections to Σ of higher order in n_{imp} [virial expansion]

- we ignore them in dilute limit

- finally let's evaluate $T(\omega)$ to $O(J^2)$

$$T(\omega) = \frac{3}{16} \frac{J^2}{V} \sum_{\vec{k}} \frac{1}{\omega - \epsilon_{\vec{k}} + i\eta} \quad \text{(related GF)}$$

Re T just shifts fermi energy a bit [$O(n_{imp})$] and leads to other unimportant correction

Im T gives scattering rate

$$\text{Im } T = -\frac{3}{16} J^2 \pi \int \frac{d^3k}{(2\pi)^3} \delta[\omega - \epsilon_{\vec{k}}]$$

$$= -\frac{3}{16} J^2 \pi \nu(\omega)$$

$$\approx -\frac{3}{16} J^2 \pi \nu \quad \text{at } \omega \to 0$$

$$\text{Im } \Sigma = -\frac{3}{16} \nu J^2 \pi n_{imp} = -\frac{1}{2\tau}$$

- NB - has correct sign - pole of Gret in negative half-plane

$$\frac{1}{\tau} = \frac{3}{2}\pi J^2 \nu \cdot n_{imp}$$

- plug into Kubo formula (or Boltzmann equation result in AcM)

$$\rho = \frac{3}{2e^2 v_F^2 \nu} \times \frac{1}{\tau} = \frac{9\pi}{16 e^2 v_F^2} J^2 n_{imp}$$

- explicit calculation $\sim O(J^3)$ gives

$$\rho = \frac{9\pi}{16 e^2 v_F^2 \nu^2}\left[\lambda_0^2 + 2\lambda_0^3 \ln\frac{D_0}{T} + \dots\right]$$

$$\sim \frac{9\pi}{16 e^2 v_F^2 \nu^2}\left[\lambda(T)^2 + \dots\right]$$

- i.e. we get RG-improved result as we expect - T provides scale at which to evaluate effective coupling

- correction is for 

- an ∞ series of diagrams corresponds to RG result

$\Rightarrow f \propto \frac{n_{mp}}{\ln^2 T/T_K}$ for $T \gg T_K$

- we must go beyond RG improved perturb theory to study $T \lesssim T_K$

- but first return to another ~~missing~~ missing Lecture next Wed 1-2:30 $\triangleq$ 12:30-2:00 ??

1/3 ~~Relevant, Ir~~ to more Complete Def^n of RG

Relevant, irrelevant & marginal operators

- so far I've done "RG light" & haven't justified dropping terms generated by RG transformation such as

$$\int d\tau \sum_{\vec{k},\vec{k}'} \bar{\psi}^{\dagger}_{\vec{k}}(\tau)\vec{\sigma}\frac{d}{d\tau}\psi_{\vec{k}'}(\tau)\cdot\vec{S}(\tau)$$

- I now want to justify this & give full def^n of RG, using Kondo model as an example

- I will first streamline notation and establish connection of Kondo model with relativistic Q.F.T.

- note that Kondo interaction only involves s-wave component of $\psi(\vec{r}) \propto \int d^2\hat{r}\ \psi(\hat{r}\, r) \propto \psi_{s,r}$

$\psi_{s,r} \propto \int d^2R\ \psi_{\hat{R}\, R}$

- in terms of s-wave component, passing to discrete limit & normalizing s-wave fermion field as $\{\psi(x), \psi^\dagger(x')\} = 2\pi\,\delta(x-x')$

[note unusual 2π, as in Shankar]

$$H_0 = v_F \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi}\, k\, \psi^\dagger(k)\psi(k)$$

- here I assumed cut-off $D_0 \ll$ band-width & linearized dispersion relation $\varepsilon = v_F(k-k_F)$, shifted $k - k_F \to k$ and let $\Lambda = \frac{D_0}{v_F}$

$$S_0 = \int_{-\infty}^{\infty} \frac{dw}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} (iw - vFk)\, \bar{\psi}(k,w)\psi(k,w)$$

$$S_{INT} = -\lambda vF \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{2\pi} \int_{-\infty}^{\infty} \frac{dw'}{2\pi}$$

$$\bar{\psi}(w,k)\, \frac{\vec{\sigma}}{2}\, \psi(w-w', k-k') \cdot \vec{S}(w')$$

where I introduced Fourier transform of impurity spin operator

- S_{INT} conserves frequency, not momentum

- if we Fourier transform back to time & space, with a 1D co-ordinate x

$$S_0 = -\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dx\; \bar{\psi}(x,\tau)\left[\frac{d}{d\tau} - iv\frac{d}{dx}\right]\psi(x,\tau)$$

- corresponds to a relativistic (Dirac, left moving only) fermion, with $c \to vF$ [obtained by unfolding $x \in [-\infty, 0]$]

$$S_{INT} = -\lambda vF \int d\tau\; \bar{\psi}(0,\tau)\, \frac{\vec{\sigma}}{2}\, \psi(0,\tau) \cdot \vec{S}(\tau)$$

- this destroys Lorentz invariance, & even spatial translation invariance

RG step reduces $\Lambda \to \frac{\Lambda}{s}$ $[s > 1]$

- reducing Λ in infinitesimal steps $s \gtrsim 1$

- after RG step

$$S_0 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda/s}^{\Lambda/s} \frac{dk}{2\pi} (i\omega - v_F k)\, \Psi(\omega, k)\, \Psi(\omega, k)$$

- it is convenient & conventional to rescale

R after reducing cut-off $\tilde{k} = k \cdot s$

to make S_0 have same form as before

$$\int_{-\Lambda/s}^{\Lambda/s} dk = \frac{1}{s} \int_{-\Lambda}^{\Lambda} d\tilde{k}$$

[will eventually drop the $\sim$]

- also rescale $\tilde{\omega} = \omega s$ so $(i\omega - v_F k)$

factor remains unchanged

$$S_0 \Rightarrow \frac{1}{s^3} \int_{-\infty}^{\infty} \frac{d\tilde{\omega}}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{d\tilde{k}}{2\pi} (i\tilde{\omega} - v_F \tilde{k})\, \Psi(\tilde{\omega}, \tilde{k})\, \Psi(\tilde{\omega}, \tilde{k})$$

- finally we rescale fields to leave S_0

invariant $\psi' = s^{-3/2}\psi$, $\bar{\psi}' = s^{-3/2}\bar{\psi}$ so

$$S_0 \to \int_{-\infty}^{\infty} \frac{d\tilde{\omega}}{2\pi} \int^{\Lambda} \frac{d\tilde{k}}{2\pi} \left(i\tilde{\omega} - v\tilde{k}\right) \bar{\psi}'(\tilde{\omega},\tilde{k})\,\psi(\tilde{\omega},\tilde{k})$$

- NB - we chose rescaling factors for fields in order to leave S_0 invariant
- this is appropriate when we are considering vicinity of free ($\lambda=0$) fixed point - which we saw was stable for $\lambda_0 < 0$
- in general we might choose some other field rescaling if we are studying vicinity of some other fixed point
- likewise, we don't always scale ω, k by same power of s - more generally

$$\tilde{k} = k s, \quad \tilde{\omega} = \omega\, s^{z}, \quad z \equiv \text{dynamical exponent}$$

- fact that dispersion relation is (approximately)

linear, $\int$, α $(i\omega - v_F k)$ dictates

this choice in this case

- rescaling of $\vec{S}(\omega)$ simply determined

by Fourier transform

$$\vec{S}(\omega) = \int_{-\infty}^{\infty} d\tau \; e^{i\omega\tau} \vec{S}(\tau)$$

$$\vec{S}(\tilde{\omega}) = \int d\tau \; e^{i \frac{\tilde{\omega}}{s} s\tau} \vec{S}(\tau)$$

$$= \frac{1}{s} \int d\tilde{\tau} \; e^{i\tilde{\omega}\tilde{\tau}} \vec{S}(\tilde{\tau})$$

$$\vec{S}(\omega) \to s \cdot \vec{S}^{new}(\tilde{\omega})$$

- Kondo interaction is invariant under this

rescaling

$$\int dk \, dk' \, d\omega \, d\omega' \; \psi(\omega')\psi(\omega+\omega')\vec{S}(\omega)$$

$$\to \int \frac{d\tilde{k} \, d\tilde{k}' \, d\tilde{\omega} \, d\tilde{\omega}'}{s^4} \; s^3 \bar{\psi}' \cdot \psi' \; s \cdot \vec{S}^{new}$$

- all factors of s cancel